

On a Simultaneous Approach to the Even and Odd Truncated Matricial Stieltjes Moment Problem I:

An α -Schur-Stieltjes-type algorithm for sequences of complex matrices

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The characterization of the solvability of matrix versions of truncated Stieltjes-Type moment problems led to the class of α -Stieltjes non-negative definite sequences of complex $q \times q$ matrices. In [21], a parametrization of this class was introduced, the so-called α -Stieltjes parametrization. The main topic of this first part of the paper is the construction of a Schur-type algorithm which produces exactly the α -Stieltjes parametrization.

1. Introduction

This paper which is divided into two parts is a direct continuation of the authors' former investigations on matrix versions of classical power moment problems and related questions (see [12, 16, 20–26]). Now our aim is to study two truncated matricial power moment problems on semi-infinite intervals. The approach is based on Schur analysis. More precisely, we will work out two interrelated versions of Schur-type algorithms similar as in our former investigations on truncated matrix moment problems of Hamburger-Type (see [24, 25]). In the first part, we will construct a Schur-type algorithm for finite sequences of complex $q \times q$ matrices. In the second part of the paper, we will construct a Schur-type algorithm for a special class of holomorphic matrix functions, which turns out to be intimately related to the truncated matricial moment problems under consideration. The starting point of studying power moment problems on semi-infinite intervals was the famous two parts memoir of Stieltjes [32, 33] where the author's investigations on questions for special continued fractions led him to the power moment problem on the interval $[0, +\infty)$. A complete theory of the treatment of power moment problems on semi-infinite intervals in the scalar case was developed by M. G. Krein in collaboration with A. A. Nudelman (see [28, Section 10], [29], [30, Ch. V]). What concerns a modern operator-theoretic treatment of the power moment problems named after Hamburger and Stieltjes and its interrelations, we refer the reader to Simon [31]. The matrix version of the classical Stieltjes moment problem was studied in Adamyan/Tkachenko [1, 2],

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Andô [3], Bolotnikov [5–7], Bolotnikov/Sakhnovich [8], Chen/Hu [9], Chen/Li [10], Dyukarev [13, 14], Dyukarev/Katsnel'son [17, 18], and Hu/Chen [27]. The considerations of this paper deal with the case of a semi-infinite interval $[\alpha, +\infty)$ and continue former work done in [16, 21, 23].

In order to properly formulate these problems, we first review some notation. Let \mathbb{C} , \mathbb{R} , \mathbb{N}_0 , and \mathbb{N} be the set of all complex numbers, the set of all real numbers, the set of all non-negative integers, and the set of all positive integers, respectively. Further, for all $\alpha, \beta \in \mathbb{R} \cup \{-\infty, +\infty\}$, let $\mathbb{Z}_{\alpha, \beta}$ be the set of all integers k for which $\alpha \leq k \leq \beta$ holds. Throughout this paper, let $p, q \in \mathbb{N}$. If \mathcal{X} is a nonempty set, then $\mathcal{X}^{p \times q}$ stands for the set of all $p \times q$ matrices each entry of which belongs to \mathcal{X} and \mathcal{X}^p is short for $\mathcal{X}^{p \times 1}$. If $k \in \mathbb{Z}$ and if $\kappa \in \mathbb{Z}_{k, +\infty} \cup \{+\infty\}$, then we denote by $\mathcal{X}_{[k, \kappa]}$ the set of all sequences $(x_j)_{j=k}^{\kappa}$ where $x_j \in \mathcal{X}$ for all $j \in \mathbb{Z}_{k, \kappa}$. If $(\mathcal{X}, \mathfrak{A})$ is a measurable space, then each countably additive mapping defined on \mathfrak{A} with values in the set $\mathbb{C}_{\geq}^{q \times q}$ of all non-negative Hermitian complex $q \times q$ matrices is called a non-negative Hermitian $q \times q$ measure on $(\mathcal{X}, \mathfrak{A})$.

Let $\alpha \in \mathbb{R}$ and let $\mathfrak{B}_{[\alpha, +\infty)}$ be the σ -algebra of all Borel subsets of $[\alpha, +\infty)$. Further, let $\mathcal{M}_{\geq}^q([\alpha, +\infty))$ be the set of all non-negative Hermitian $q \times q$ measures on $([\alpha, +\infty), \mathfrak{B}_{[\alpha, +\infty)})$ and, for all $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $\mathcal{M}_{\geq, \kappa}^q([\alpha, +\infty))$ be the set of all $\sigma \in \mathcal{M}_{\geq}^q([\alpha, +\infty))$ such that the integral

$$s_j^{(\sigma)} := \int_{[\alpha, +\infty)} t^j \sigma(dt)$$

exists for all $j \in \mathbb{Z}_{0, \kappa}$. The above-mentioned matricial moment problems can be formulated as follows:

M $[[\alpha, +\infty); (s_j)_{j=0}^{\kappa}, =]$ Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices. Parametrize the set $\mathcal{M}_{\geq}^q([\alpha, +\infty); (s_j)_{j=0}^{\kappa}, =]$ of all non-negative Hermitian measures $\sigma \in \mathcal{M}_{\geq, \kappa}^q([\alpha, +\infty))$ for which $s_j^{(\sigma)} = s_j$ is fulfilled for all $j \in \mathbb{Z}_{0, \kappa}$.

M $[[\alpha, +\infty); (s_j)_{j=0}^m, \leq]$ Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Parametrize the set $\mathcal{M}_{\geq}^q([\alpha, +\infty); (s_j)_{j=0}^m, \leq]$ of all non-negative Hermitian measures $\sigma \in \mathcal{M}_{\geq, m}^q([\alpha, +\infty))$ for which $s_m - s_m^{(\sigma)}$ is non-negative Hermitian and, in the case $m \geq 1$, moreover $s_j^{(\sigma)} = s_j$ is fulfilled for all $j \in \mathbb{Z}_{0, m-1}$.

The particular case $\alpha = 0$ is treated by several authors. Parametrizations of the set $\mathcal{M}_{\geq}^q[[0, +\infty); (s_j)_{j=0}^m, \leq]$ are given, under a certain non-degeneracy condition, in [15]. In the general case, descriptions of $\mathcal{M}_{\geq}^q[[0, +\infty); (s_j)_{j=0}^m, \leq]$ are stated by Bolotnikov [5], [6, Theorem 1.5] and by Chen/Hu in [9, Theorem 2.4], whereas descriptions of the set $\mathcal{M}_{\geq}^q[[0, +\infty); (s_j)_{j=0}^m, =]$ are given by Hu/Chen in [27, Theorem 4.1, Lemmas 2.3 and 2.4]. Moreover, for arbitrary real numbers α , the cases $\mathcal{M}_{\geq}^q[[\alpha, +\infty); (s_j)_{j=0}^m, =] \neq \emptyset$ and $\mathcal{M}_{\geq}^q[[\alpha, +\infty); (s_j)_{j=0}^m, \leq] \neq \emptyset$ are characterized in [16, Theorems 1.3 and 1.4].

Before we explain these criteria of solvability, we introduce certain sets of sequences of complex $q \times q$ matrices which are determined by the properties of particular Hankel

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matrices built of them. For all $n \in \mathbb{N}_0$, let $\mathcal{H}_{q,2n}^{\geq}$ (resp. $\mathcal{H}_{q,2n}^{>}$) be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices such that the block Hankel matrix

$$H_n := [s_{j+k}]_{j,k=0}^n \quad (1.1)$$

is non-negative Hermitian (resp. positive Hermitian). Furthermore, let $\mathcal{H}_{q,\infty}^{\geq}$ (resp. $\mathcal{H}_{q,\infty}^{>}$) be the set of all sequences $(s_j)_{j=0}^{\infty}$ of complex $q \times q$ matrices such that for all $n \in \mathbb{N}_0$ the sequence $(s_j)_{j=0}^{2n}$ belongs to $\mathcal{H}_{q,2n}^{\geq}$ (resp. $\mathcal{H}_{q,2n}^{>}$). For all $n \in \mathbb{N}_0$, let $\mathcal{H}_{q,2n}^{\geq,e}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices for which there exist complex $q \times q$ matrices s_{2n+1} and s_{2n+2} such that $(s_j)_{j=0}^{2(n+1)}$ belongs to $\mathcal{H}_{q,2(n+1)}^{\geq}$. Furthermore, for all $n \in \mathbb{N}_0$, we will use $\mathcal{H}_{q,2n+1}^{\geq,e}$ to denote the set of all sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices for which there exists a complex $q \times q$ matrix s_{2n+2} such that $(s_j)_{j=0}^{2(n+1)}$ belongs to $\mathcal{H}_{q,2(n+1)}^{\geq}$. Let $\mathcal{H}_{q,\infty}^{\geq,e} := \mathcal{H}_{q,\infty}^{\geq}$. If $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, then we call a sequence $(s_j)_{j=0}^{2\kappa}$ of complex $q \times q$ matrices *Hankel non-negative definite* (resp. *Hankel positive definite*) if it belongs to $\mathcal{H}_{q,2\kappa}^{\geq}$ (resp. $\mathcal{H}_{q,2\kappa}^{>}$), and we call a sequence $(s_j)_{j=0}^{\kappa}$ of complex $q \times q$ matrices *Hankel non-negative definite extendable* if it belongs to $\mathcal{H}_{q,\kappa}^{\geq,e}$.

Besides the just introduced classes of sequences of complex $q \times q$ matrices we need analogous classes of sequences of complex $q \times q$ matrices which take into account the influence of the prescribed number $\alpha \in \mathbb{R}$: Let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Then, for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, we introduce the block Hankel matrix

$$K_n := [s_{j+k+1}]_{j,k=0}^n. \quad (1.2)$$

If $\kappa \geq 1$, then, for all $\alpha \in \mathbb{C}$, let the sequence $(s_{\alpha \triangleright j})_{j=0}^{\kappa-1}$ be given by

$$s_{\alpha \triangleright j} := -\alpha s_j + s_{j+1} \quad (1.3)$$

for all $j \in \mathbb{Z}_{0,\kappa-1}$. Then $(s_{\alpha \triangleright j})_{j=0}^{\kappa-1}$ is called the *sequence generated from $(s_j)_{j=0}^{\kappa}$ by right-sided α -shifting*. (An analogous left-sided version is discussed in [21, Definition 2.1].)

Let $\alpha \in \mathbb{R}$. Now we will introduce several classes of finite or infinite sequences of complex $q \times q$ matrices which are characterized by the sequences $(s_j)_{j=0}^{\kappa}$ and $(s_{\alpha \triangleright j})_{j=0}^{\kappa-1}$. Let $\mathcal{K}_{q,0,\alpha}^{\geq} := \mathcal{H}_{q,0}^{\geq}$, and, for all $n \in \mathbb{N}$, let $\mathcal{K}_{q,2n,\alpha}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices for which the block Hankel matrices H_n and $-\alpha H_{n-1} + K_{n-1}$ are both non-negative Hermitian, i. e.,

$$\mathcal{K}_{q,2n,\alpha}^{\geq} := \left\{ (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} \mid (s_{\alpha \triangleright j})_{j=0}^{2(n-1)} \in \mathcal{H}_{q,2(n-1)}^{\geq} \right\}. \quad (1.4)$$

Furthermore, for all $n \in \mathbb{N}_0$, let $\mathcal{K}_{q,2n+1,\alpha}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices for which the block Hankel matrices H_n and $-\alpha H_n + K_n$ are both non-negative Hermitian, i. e.,

$$\mathcal{K}_{q,2n+1,\alpha}^{\geq} := \left\{ (s_j)_{j=0}^{2n+1} \in (\mathbb{C}^{q \times q})_{[0,2n+1]} \mid \{(s_j)_{j=0}^{2n}, (s_{\alpha \triangleright j})_{j=0}^{2n}\} \subseteq \mathcal{H}_{q,2n}^{\geq} \right\}. \quad (1.5)$$

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Formulas (1.4) and (1.5) show that the sets $\mathcal{K}_{q,2n,\alpha}^{\geq}$ and $\mathcal{K}_{q,2n+1,\alpha}^{\geq}$ are determined by two conditions. The condition $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$ ensures that a particular Hamburger moment problem associated with the sequence $(s_j)_{j=0}^{2n}$ is solvable (see, e.g. [12, Theorem 4.16]). The second condition $(s_{\alpha \triangleright j})_{j=0}^{2(n-1)} \in \mathcal{H}_{q,2(n-1)}^{\geq}$ (resp. $(s_{\alpha \triangleright j})_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$) controls that the original sequences $(s_j)_{j=0}^{2n}$ and $(s_j)_{j=0}^{2n+1}$ are well adapted to the interval $[\alpha, +\infty)$. Let $m \in \mathbb{N}_0$. Then, let $\mathcal{K}_{q,m,\alpha}^{\geq,e}$ be the set of all sequences $(s_j)_{j=0}^m$ of complex $q \times q$ matrices for which there exists a complex $q \times q$ matrix s_{m+1} such that $(s_j)_{j=0}^{m+1}$ belongs to $\mathcal{K}_{q,m+1,\alpha}^{\geq}$. We call a sequence $(s_j)_{j=0}^m$ of complex $q \times q$ matrices *α -Stieltjes non-negative definite* if it belongs to $\mathcal{K}_{q,m,\alpha}^{\geq}$ and *α -Stieltjes non-negative definite extendable* if it belongs to $\mathcal{K}_{q,m,\alpha}^{\geq,e}$.

Let us recall solvability criterions for the Problems $\mathbf{M}[[\alpha, +\infty); (s_j)_{j=0}^m, =]$ and $\mathbf{M}[[\alpha, +\infty); (s_j)_{j=0}^m, \leq]$:

Theorem 1.1 ([16, Theorems 1.3 and 1.4]). *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m$ be a sequence of complex $q \times q$ matrices. Then:*

- (a) $\mathcal{M}_{\geq}^q[[\alpha, +\infty); (s_j)_{j=0}^m, =] \neq \emptyset$ if and only if $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,e}$.
- (b) $\mathcal{M}_{\geq}^q[[\alpha, +\infty); (s_j)_{j=0}^m, \leq] \neq \emptyset$ if and only if $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq}$.

Theorem 1.1 provides a first impression on the importance of the classes of α -Stieltjes non-negative definite sequences and α -Stieltjes non-negative definite extendable sequences in the framework of truncated matricial Stieltjes-type moment problems related to the interval $[\alpha, +\infty)$. The role of these classes is by far not restricted to these solvability conditions. A thorough study of these classes forms an essential cornerstone in our approach to describe the solution sets of both moment problems via Schur analysis methods. For this reason, we extend our former investigations (see [16, 21]) on α -Stieltjes non-negative definite sequences and remarkable subclasses by constructing a special Schur-type algorithm for finite sequences of complex $p \times q$ matrices. It should be mentioned that we will consider the matricial moment problems under consideration in the most general case. The so-called non-degenerate case is connected to the class of so-called α -Stieltjes positive definite sequences (see Section 2).

In this paper, we construct a Schur-type algorithm for sequences belonging to the class $\mathcal{K}_{q,m,\alpha}^{\geq}$. It turns out that this algorithm produces exactly the α -Stieltjes parametrization of such sequences which was introduced in [21]. Particular attention is paid to several natural subclasses of $\mathcal{K}_{q,m,\alpha}^{\geq}$. We indicate that the Schur-type algorithm under consideration is used in [19] to give a parametrization of the solution set of Problem $\mathbf{M}[[\alpha, +\infty); (s_j)_{j=0}^m, =]$.

This paper is organized as follows: In Section 2, we introduce the classes of sequences of complex $p \times q$ matrices under consideration and recall the notion of α -Stieltjes parametrization. Similarly as in [25], the algorithm we are going to construct will be based on the concept of reciprocal sequences. For this reason, we summarize in Section 3 some basic facts on the arithmetics of reciprocal sequences. Furthermore, we introduce the classes of first term dominant sequences and nearly first term dominant sequences of complex $p \times q$ matrices (see Definitions 3.3 and 3.7, respectively). The main result

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of Section 3 is Proposition 3.8, which contains several interrelations between important subclasses of $\mathcal{K}_{q,\kappa,\alpha}^{\geq}$ and the classes of (nearly) first term dominant sequences. Section 4 marks an essential stage for the construction of our algorithm. Let us first mention what happened in [25]. There we started with a sequence $(s_j)_{j=0}^{2n}$ from $\mathbb{C}^{p \times q}$ and considered its corresponding reciprocal sequence. Now we have to ensure that our construction takes into account the influence of α from the very beginning. In view of the definition of the class $\mathcal{K}_{q,\kappa,\alpha}^{\geq}$, we should use, in some way, the operation of right-sided α -shifting of sequences given via (1.3). If $(s_j)_{j=0}^{\kappa}$ is a sequence from $\mathbb{C}^{p \times q}$, then the right-sided α -shifting $(s_{\alpha \triangleright j})_{j=0}^{\kappa-1}$ of $(s_j)_{j=0}^{\kappa}$ is a sequence of length $\kappa - 1$. Since our construction should start with a sequence of length κ , we consider a slight modification of $(s_{\alpha \triangleright j})_{j=0}^{\kappa-1}$. This leads us to the reciprocal sequence $(s_j^{[\sharp, \alpha]})_{j=0}^{\kappa}$ corresponding to the $[+, \alpha]$ -transform $(s_j^{[+, \alpha]})_{j=0}^{\kappa}$. Section 5 plays an analogous role as Section 6 in [25]. The main goal is to derive identities between various block Hankel matrices built from the sequences $(s_j^{[+, \alpha]})_{j=0}^{\kappa}$ and $(s_j^{[\sharp, \alpha]})_{j=0}^{\kappa}$ (see Theorems 5.6 and 5.8). Section 6 should be compared with [25, Section 7], where we considered a sequence $(s_j)_{j=0}^{2n}$ from $\mathbb{C}^{q \times q}$ together with its corresponding reciprocal sequence $(s_j^{\sharp})_{j=0}^{2n}$. From [25, Propositions 8.25 and 8.26] we see that the membership of $(s_j)_{j=0}^{2n}$ to the class of Hankel non-negative definite sequences and some of its distinguished subclasses will be preserved for the sequence $(-s_{j+2}^{\sharp})_{j=0}^{2(n-1)}$. Section 6 is aimed to realize a construction suitable for the purposes of this paper, where a one-step algorithm is used. The main result of Section 6 is Proposition 6.13, which indicated that the membership of $(s_j)_{j=0}^{\kappa}$ to the class of α -Stieltjes non-negative definite sequences and its prominent subclasses is preserved by the sequence $(-s_{j+1}^{[\sharp, \alpha]})_{j=0}^{\kappa-1}$. In Section 7, the shortened negative reciprocal sequence $(-s_{j+1}^{[\sharp, \alpha]})_{j=0}^{\kappa-1}$ will be replaced by a slightly modified sequence. In this way, we consider a transformation which ensures that an arbitrary sequence $(s_j)_{j=0}^{\kappa}$ belonging to $\mathcal{K}_{q,\kappa,\alpha}^{\geq}$ and $\mathcal{K}_{q,\kappa,\alpha}^{\geq, e}$, respectively, is transformed into a sequence $(s_j^{[1, \alpha]})_{j=0}^{\kappa-1}$ belonging to $\mathcal{K}_{q,\kappa-1,\alpha}^{\geq}$ and $\mathcal{K}_{q,\kappa-1,\alpha}^{\geq, e}$, respectively (see Theorem 7.21). The iteration of this transformation, discussed in Section 7, leads us to a particular algorithm for finite or infinite sequences of complex $p \times q$ matrices (see Section 8). We show that this algorithm preserves the membership of a sequence to the class of α -Stieltjes non-negative definite sequences and its prominent subclasses (see Theorem 8.10). In Section 9, we prove that this algorithm produces exactly the right α -Stieltjes parametrization. Central results of the paper are Theorems 9.12 and 9.15, which contain explicit constructions of the right α -Stieltjes parametrization of a sequence belonging to one of the classes $\mathcal{K}_{q,m,\alpha}^{\geq}$ or $\mathcal{K}_{q,m,\alpha}^{\geq, e}$. Theorems 9.25 and 9.26 are focused on the determination of the right α -Stieltjes parametrization of the described transform of a sequence belonging to one of the classes $\mathcal{K}_{q,\kappa,\alpha}^{\geq}$ or $\mathcal{K}_{q,\kappa,\alpha}^{\geq, e}$. These results indicate that the right α -Stieltjes parametrization can be interpreted as a Schur-type parametrization. In Section 10, we study several aspects of some inverse transformation.

As already mentioned above, the main goal of this paper is to construct a special Schur-type algorithm for finite or infinite sequences of complex $p \times q$ matrices. We are guided by our former experiences in constructing another version of Schur-type algorithm for

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finite or infinite sequences of complex $p \times q$ matrices in [25], which is a two-step algorithm. In this paper, we discuss a one-step algorithm. In the case $\alpha = 0$, such an algorithm was developed already in [9]. An essential feature of the Schur-type algorithm constructed in [25] is its intimate connection to the canonical Hankel parametrization of Hankel non-negative definite sequences. We will demonstrate that the Schur-type algorithm discussed in this paper has a similar connection to the right α -Stieltjes parametrization of α -Stieltjes non-negative definite sequences. Although the basic techniques in this paper are strongly influenced by the investigations in [25], it should be remarked that the situation is now more complicated. Indeed, now we have to control the interrelation between two Hankel non-negative definite sequences from $\mathbb{C}^{q \times q}$. This interplay is governed by a real number α . For this reason, we will meet rather new effects in comparison with [25]. In order to manage them, a substantial refinement of the techniques used in [25] is necessary. In this way, we are forced to introduce various constructions and transforms depending on α .

2. Right α -Stieltjes parametrization

In this section, we recall the concept of right α -Stieltjes parametrization of finite or infinite sequences which was developed in [16, 21].

Remark 2.1. Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq}$ (resp. $\mathcal{K}_{q,m,\alpha}^{\geq,e}$). Then we easily see that $(s_j)_{j=0}^l \in \mathcal{K}_{q,l,\alpha}^{\geq}$ (resp. $\mathcal{K}_{q,l,\alpha}^{\geq,e}$) for all $l \in \mathbb{Z}_{0,m}$.

Let $\alpha \in \mathbb{R}$. In view of Remark 2.1, let $\mathcal{K}_{q,\infty,\alpha}^{\geq}$ be the set of all sequences $(s_j)_{j=0}^\infty$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq}$ for all $m \in \mathbb{N}_0$. Further, let $\mathcal{K}_{q,\infty,\alpha}^{\geq,e} := \mathcal{K}_{q,\infty,\alpha}^{\geq}$. Obviously, for all $n \in \mathbb{N}$, we have

$$\mathcal{K}_{q,2n,\alpha}^{\geq,e} = \left\{ (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} \mid (s_{\alpha \triangleright j})_{j=0}^{2n-1} \in \mathcal{H}_{q,2n-1}^{\geq,e} \right\}$$

and, for all $n \in \mathbb{N}_0$, furthermore

$$\mathcal{K}_{q,2n+1,\alpha}^{\geq,e} = \left\{ (s_j)_{j=0}^{2n+1} \in \mathcal{H}_{q,2n+1}^{\geq,e} \mid (s_{\alpha \triangleright j})_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} \right\}. \quad (2.1)$$

Let $\mathcal{K}_{q,0,\alpha}^> := \mathcal{H}_{q,0}^>$, and, for all $n \in \mathbb{N}$, let $\mathcal{K}_{q,2n,\alpha}^>$ be the set of all sequences $(s_j)_{j=0}^{2n}$ of complex $q \times q$ matrices for which the block Hankel matrices H_n and $-\alpha H_{n-1} + K_{n-1}$ are positive Hermitian, i. e., $\mathcal{K}_{q,2n,\alpha}^> := \{(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^> \mid (s_{\alpha \triangleright j})_{j=0}^{2(n-1)} \in \mathcal{H}_{q,2(n-1)}^>\}$. Furthermore, for all $n \in \mathbb{N}_0$, let $\mathcal{K}_{q,2n+1,\alpha}^>$ be the set of all sequences $(s_j)_{j=0}^{2n+1}$ of complex $q \times q$ matrices for which the block Hankel matrices H_n and $-\alpha H_n + K_n$ are positive Hermitian, i. e.,

$$\mathcal{K}_{q,2n+1,\alpha}^> := \left\{ (s_j)_{j=0}^{2n+1} \in (\mathbb{C}^{q \times q})_{[0,2n+1]} \mid \{(s_j)_{j=0}^{2n}, (s_{\alpha \triangleright j})_{j=0}^{2n}\} \subseteq \mathcal{H}_{q,2n}^> \right\}.$$

Let $\mathcal{K}_{q,\infty,\alpha}^>$ be the set of all sequences $(s_j)_{j=0}^\infty$ of complex $q \times q$ matrices such that $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^>$ for all $m \in \mathbb{N}_0$. For all $n \in \mathbb{N}_0$, let

$$\mathcal{H}_{q,2n}^{\geq,cd} := \left\{ (s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq} \mid L_n = 0_{q \times q} \right\}, \quad (2.2)$$

$$\mathcal{K}_{q,2n,\alpha}^{\geq,cd} := \mathcal{K}_{q,2n,\alpha}^{\geq} \cap \mathcal{H}_{q,2n}^{\geq,cd}, \quad (2.3)$$

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and let

$$\mathcal{K}_{q,2n+1,\alpha}^{\geq,\text{cd}} := \left\{ (s_j)_{j=0}^{2n+1} \in \mathcal{K}_{q,2n+1,\alpha}^{\geq} \mid (s_{\alpha \triangleright j})_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\text{cd}} \right\}. \quad (2.4)$$

Furthermore, let $\mathcal{K}_{q,\infty,\alpha}^{\geq,\text{cd}}$ be the set of all sequences $(s_j)_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\geq}$ for which there exists a number $m \in \mathbb{N}_0$ such that $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,\text{cd}}$. For all $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and all $m \in \mathbb{Z}_{0,\kappa}$, let

$$\mathcal{K}_{q,\kappa,\alpha}^{\geq,\text{cd},m} := \left\{ (s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq} \mid (s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,\text{cd}} \right\}. \quad (2.5)$$

Obviously,

$$\bigcup_{m=0}^{\infty} \mathcal{K}_{q,\infty,\alpha}^{\geq,\text{cd},m} = \mathcal{K}_{q,\infty,\alpha}^{\geq,\text{cd}}. \quad (2.6)$$

Remark 2.2. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq}$ (resp. $\mathcal{K}_{q,\kappa,\alpha}^{\geq,\text{e}}$), and let $A \in \mathbb{C}^{q \times p}$. Then $(A^* s_j A)_{j=0}^{\kappa} \in \mathcal{K}_{p,\kappa,\alpha}^{\geq}$ (resp. $\mathcal{K}_{p,\kappa,\alpha}^{\geq,\text{e}}$).

We write $\mathbb{C}_H^{q \times q}$ for the set of all Hermitian complex $q \times q$ matrices and we use the Löwner semi-ordering in $\mathbb{C}_H^{q \times q}$, i. e., we write $A \geq B$ (resp. $A > B$) in order to indicate that A and B are Hermitian complex matrices such that $A - B$ is non-negative (resp. positive) Hermitian. We denote by $\mathcal{N}(A)$ and $\mathcal{R}(A)$ the null space and the column space of a complex matrix A , respectively. For all $\beta \in \mathbb{R}$, let $\lfloor \beta \rfloor := \max\{k \in \mathbb{Z} \mid k \leq \beta\}$ be the largest integer not greater than β . It is useful to state the following technical results:

Lemma 2.3 ([21, Lemma 2.9]). *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq}$. Then:*

- (a) $s_j \in \mathbb{C}_H^{q \times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$ and $s_{\alpha \triangleright j} \in \mathbb{C}_H^{q \times q}$ for all $j \in \mathbb{Z}_{0,\kappa-1}$.
- (b) $s_{2k} \in \mathbb{C}_{\geq}^{q \times q}$ for all $k \in \mathbb{Z}_{0,\frac{\kappa}{2}}$ and $s_{\alpha \triangleright 2k} \in \mathbb{C}_{\geq}^{q \times q}$ for all $k \in \mathbb{Z}_{0,\frac{\kappa-1}{2}}$.
- (c) $\mathcal{N}(s_{2k}) \subseteq \mathcal{N}(s_j)$ and $\mathcal{R}(s_j) \subseteq \mathcal{R}(s_{2k})$ for all $k \in \mathbb{Z}_{0,\frac{\kappa}{2}}$ and all $j \in \mathbb{Z}_{2k,2\lfloor \frac{\kappa}{2} \rfloor-1}$.
- (d) $\mathcal{N}(s_{\alpha \triangleright 2k}) \subseteq \mathcal{N}(s_{\alpha \triangleright j})$ and $\mathcal{R}(s_{\alpha \triangleright j}) \subseteq \mathcal{R}(s_{\alpha \triangleright 2k})$ for all $k \in \mathbb{Z}_{0,\frac{\kappa-1}{2}}$ and all $j \in \mathbb{Z}_{2k,2\lfloor \frac{\kappa-1}{2} \rfloor-1}$.

Lemma 2.4 (see [16, Lemmas 4.7 and 4.11]). *Let $\alpha \in \mathbb{R}$ and let $n \in \mathbb{N}_0$. Then $\mathcal{K}_{q,2n,\alpha}^{\geq,\text{e}} \subseteq \mathcal{H}_{q,2n}^{\geq,\text{e}}$. Furthermore, if $(s_j)_{j=0}^{2n+1} \in \mathcal{K}_{q,2n+1,\alpha}^{\geq,\text{e}}$, then $(s_{\alpha \triangleright j})_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq,\text{e}}$.*

In the sequel, we will need the Moore-Penrose inverse A^\dagger of a complex $p \times q$ matrix A . In order to formulate some essential observations on the above introduced sets, it is useful to introduce now some further constructions of matrices, which will play a central role in the following. If $n \in \mathbb{N}$, if $(p_j)_{j=1}^n$ is a sequence of positive integers, and if $A_j \in \mathbb{C}^{p_j \times q}$ for all $j \in \mathbb{Z}_{1,n}$, then let

$$\text{col}(A_j)_{j=1}^n := \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}.$$

2. Right α -Stieltjes parametrization

Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. We will associate with $(s_j)_{j=0}^\kappa$ several matrices which we will often need in our subsequent considerations: For all $l, m \in \mathbb{N}_0$ with $l \leq m \leq \kappa$, let

$$y_{l,m}^{(s)} := \text{col}(s_j)_{j=l}^m \quad \text{and} \quad z_{l,m}^{(s)} := [s_l, s_{l+1}, \dots, s_m]. \quad (2.7)$$

For all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let

$$H_n^{(s)} := [s_{j+k}]_{j,k=0}^n, \quad (2.8)$$

for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, let

$$K_n^{(s)} := [s_{j+k+1}]_{j,k=0}^n, \quad (2.9)$$

and, for all $n \in \mathbb{N}_0$ with $2n+2 \leq \kappa$, let

$$G_n^{(s)} := [s_{j+k+2}]_{j,k=0}^n. \quad (2.10)$$

Let

$$L_0^{(s)} := s_0, \quad L_n^{(s)} := s_{2n} - z_{n,2n-1}^{(s)} (H_{n-1}^{(s)})^\dagger y_{n,2n-1}^{(s)}, \quad (2.11)$$

and let

$$\mathbb{L}_n^{(s)} := G_{n-1}^{(s)} - y_{1,n}^{(s)} s_0^\dagger z_{1,n}^{(s)} \quad (2.12)$$

for all $n \in \mathbb{N}$ with $2n \leq \kappa$. Let

$$\Theta_0^{(s)} := 0_{p \times q} \quad \text{and} \quad \Theta_n^{(s)} := z_{n,2n-1}^{(s)} (H_{n-1}^{(s)})^\dagger y_{n,2n-1}^{(s)} \quad (2.13)$$

for all $n \in \mathbb{N}$ with $2n-1 \leq \kappa$, where $0_{p \times q}$ denotes the zero matrix in $\mathbb{C}^{p \times q}$. In situations in which it is clear which sequence $(s_j)_{j=0}^\kappa$ of complex matrices is meant, we will write $y_{l,m}$, $z_{l,m}$, H_n , K_n , G_n , L_n , \mathbb{L}_n , and Θ_n instead of $y_{j,k}^{(s)}$, $z_{j,k}^{(s)}$, $H_n^{(s)}$, $K_n^{(s)}$, $G_n^{(s)}$, $L_n^{(s)}$, $\mathbb{L}_n^{(s)}$, and $\Theta_n^{(s)}$, respectively.

Let $\alpha \in \mathbb{C}$ and let $\kappa \geq 1$. Then the sequence $(v_j)_{j=0}^{\kappa-1}$ given by $v_j := s_{\alpha \triangleright j}$ and (1.3) for all $j \in \mathbb{Z}_{0,\kappa-1}$ plays a key role in our subsequent considerations. We define

$$\Theta_{\alpha \triangleright n} := \Theta_n^{(v)} \quad (2.14)$$

for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$,

$$H_{\alpha \triangleright n} := H_n^{(v)} \quad (2.15)$$

and

$$L_{\alpha \triangleright n} := L_n^{(v)} \quad (2.16)$$

for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$,

$$K_{\alpha \triangleright n} := K_n^{(v)}$$

for all $n \in \mathbb{N}_0$ with $2n+2 \leq \kappa$, and

$$y_{\alpha \triangleright l,m} := y_{l,m}^{(v)} \quad \text{and} \quad z_{\alpha \triangleright l,m} := z_{l,m}^{(v)} \quad (2.17)$$

2. Right α -Stieltjes parametrization

for all $l, m \in \mathbb{N}_0$ with $l \leq m \leq \kappa$. In view of (1.1), (1.2), (1.3), and (2.15), then

$$-\alpha H_n + K_n = H_{\alpha \triangleright n} \quad (2.18)$$

for all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$.

Let us recall useful characterizations of the set $\mathcal{K}_{q,m,\alpha}^{\geq,e}$:

Lemma 2.5 ([16, Lemma 4.15]). *Let $\alpha \in \mathbb{R}$, let $n \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{2n+1} \in \mathcal{K}_{q,2n+1,\alpha}^{\geq}$. Then $(s_j)_{j=0}^{2n+1} \in \mathcal{K}_{q,2n+1,\alpha}^{\geq,e}$ if and only if $\mathcal{N}(L_n) \subseteq \mathcal{N}(L_{\alpha \triangleright n})$.*

Lemma 2.6 ([16, Lemma 4.16]). *Let $\alpha \in \mathbb{R}$, let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n} \in \mathcal{K}_{q,2n,\alpha}^{\geq}$. Then $(s_j)_{j=0}^{2n} \in \mathcal{K}_{q,2n,\alpha}^{\geq,e}$ if and only if $\mathcal{N}(L_{\alpha \triangleright n-1}) \subseteq \mathcal{N}(L_n)$.*

Our current focus on matricial moment problems for intervals of the type $[\alpha, +\infty)$ or $(-\infty, \alpha]$, where α is an arbitrary real number, motivates us to look for corresponding one-sided α -analogues of the canonical Hankel parametrization of sequences of complex $p \times q$ matrices. Our above considerations show that, for reasons of symmetry, we can mainly concentrate on the right case. In this case, we use the matrices defined in (2.11) and (2.16) to introduce the notion in Definition 2.8 below, which will turn out to be one of the central objects of this paper.

Remark 2.7. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then one can easily see that there is a unique sequence $(Q_j)_{j=0}^\kappa$ of complex $p \times q$ matrices such that $s_{2k} = \Theta_k + Q_{2k}$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$ and $s_{2k+1} = \alpha s_{2k} + \Theta_{\alpha \triangleright k} + Q_{2k+1}$ for all $k \in \mathbb{N}_0$ with $2k+1 \leq \kappa$. In particular, we see that $Q_{2k} = L_k$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$ and moreover $Q_{2k+1} = L_{\alpha \triangleright k}$ for all $k \in \mathbb{N}_0$ with $2k+1 \leq \kappa$.

Remark 2.7 leads us to the following notion, which was introduced in [21].

Definition 2.8. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then the sequence $(Q_j)_{j=0}^\kappa$ given by $Q_{2k} := L_k$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$ and by $Q_{2k+1} := L_{\alpha \triangleright k}$ for all $k \in \mathbb{N}_0$ with $2k+1 \leq \kappa$ is called the *right α -Stieltjes parametrization of $(s_j)_{j=0}^\kappa$* . In the case $\alpha = 0$, the sequence $(Q_j)_{j=0}^\kappa$ is simply called the *right Stieltjes parametrization of $(s_j)_{j=0}^\kappa$* .

Remark 2.9. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(Q_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then it can be immediately checked by induction that there is a unique sequence $(s_j)_{j=0}^\kappa$ from $\mathbb{C}^{p \times q}$ such that $(Q_j)_{j=0}^\kappa$ is the right α -Stieltjes parametrization of $(s_j)_{j=0}^\kappa$, namely the sequence $(s_j)_{j=0}^\kappa$ recursively given by $s_{2k} = \Theta_k + Q_{2k}$ for all $k \in \mathbb{N}_0$ with $2k \leq \kappa$ and $s_{2k+1} = \alpha s_{2k} + \Theta_{\alpha \triangleright k} + Q_{2k+1}$ for all $k \in \mathbb{N}_0$ with $2k+1 \leq \kappa$.

Remark 2.10. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Denote by $(Q_j)_{j=0}^\kappa$ the right α -Stieltjes parametrization of $(s_j)_{j=0}^\kappa$. For all $m \in \mathbb{Z}_{0,\kappa}$, then $(Q_j)_{j=0}^m$ is exactly the right α -Stieltjes parametrization of $(s_j)_{j=0}^m$.

The following result shows that the membership of a sequence $(s_j)_{j=0}^\kappa$ of complex $q \times q$ matrices to one of the classes $\mathcal{K}_{q,\kappa,\alpha}^{\geq}$ and $\mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$ can be effectively characterized in terms of its right α -Stieltjes parametrization.

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Theorem 2.11 (see [21, Theorem 4.12]). *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices with α -Stieltjes parametrization $(Q_j)_{j=0}^\kappa$. Then:*

- (a) *The sequence $(s_j)_{j=0}^\kappa$ belongs to $\mathcal{K}_{q,\kappa,\alpha}^>$ if and only if $Q_j \in \mathbb{C}_{>}^{q \times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$ and, in the case $\kappa \geq 2$, furthermore $\mathcal{N}(Q_j) \subseteq \mathcal{N}(Q_{j+1})$ for all $j \in \mathbb{Z}_{0,\kappa-2}$.*
- (b) *The sequence $(s_j)_{j=0}^\kappa$ belongs to $\mathcal{K}_{q,\kappa,\alpha}^{>e}$ if and only if $Q_j \in \mathbb{C}_{>}^{q \times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$ and, in the case $\kappa \geq 1$, furthermore $\mathcal{N}(Q_j) \subseteq \mathcal{N}(Q_{j+1})$ for all $j \in \mathbb{Z}_{0,\kappa-1}$.*
- (c) *The sequence $(s_j)_{j=0}^\kappa$ belongs to $\mathcal{K}_{q,\kappa,\alpha}^>$ if and only if $Q_j \in \mathbb{C}_{>}^{q \times q}$ for all $j \in \mathbb{Z}_{0,\kappa}$.*

At the end of this section, it should be mentioned that, in the case $\alpha = 0$, there is a particular parametrization of sequences belonging to the class $\mathcal{K}_{q,\infty,\alpha}^>$ which originates in Yu. M. Dyukarev's paper [15], where the moment problem $\mathbf{M}[[0, +\infty); (s_j)_{j=0}^\infty, =]$ was studied. One of his main results is a generalization of a classical criterion due to Stieltjes [32, 33] for the indeterminacy of this moment problem. Yu. M. Dyukarev had to look for a convenient matricial generalization of the parameter sequence which Stieltjes obtained from the considerations of particular continued fractions associated with the sequence $(s_j)_{j=0}^\infty$. In this way, Yu. M. Dyukarev found an interesting inner parametrization of sequences belonging to $\mathcal{K}_{q,\infty,\alpha}^>$. This parametrization was reconsidered by the authors in [23, Definition 8.2]. The main theme of [23, Section 8] was to state interrelations between the Dyukarev-Stieltjes parametrization and our right 0-Stieltjes parametrization introduced in Definition 2.8.

3. The concept of reciprocal sequences

The concept used in this section of constructing a special transformation for finite and infinite sequences of complex $p \times q$ matrices is presented in [26]. The cited paper deals with the question of invertibility as it applies to finite and infinite sequences of complex $p \times q$ matrices. Two special notions, introduced there will prove to be of particular importance throughout this paper. That's why we recall the definitions.

Definition 3.1 ([26, Definition 4.13]). Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. The sequence $(s_j^\sharp)_{j=0}^\kappa$ given by $s_0^\sharp := s_0^\dagger$ and $s_j^\sharp := -s_0^\dagger \sum_{l=0}^{j-1} s_{j-l} s_l^\sharp$ for all $j \in \mathbb{Z}_{1,\kappa}$ is called the *reciprocal sequence corresponding to $(s_j)_{j=0}^\kappa$* .

Remark 3.2. Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices with reciprocal sequence $(s_j^\sharp)_{j=0}^\kappa$. For all $m \in \mathbb{Z}_{0,\kappa}$ then $(s_j^\sharp)_{j=0}^m$ is the reciprocal sequence corresponding to $(s_j)_{j=0}^m$.

Definition 3.3 ([26, Definition 4.3]). Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. We then say that $(s_j)_{j=0}^\kappa$ is *first term dominant* if $\mathcal{N}(s_0) \subseteq \bigcap_{j=0}^\kappa \mathcal{N}(s_j)$ and $\bigcup_{j=0}^\kappa \mathcal{R}(s_j) \subseteq \mathcal{R}(s_0)$. The set of all first term dominant sequences $(s_j)_{j=0}^\kappa$ of complex $p \times q$ matrices will be denoted by $\mathcal{D}_{p \times q, \kappa}$.

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Remark 3.4. Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then $(s_j)_{j=0}^\kappa \in \mathcal{D}_{p \times q, \kappa}$ if and only if for $j \in \mathbb{Z}_{0, \kappa}$ the equations $s_j s_0^\dagger s_0 = s_j$ and $s_0 s_0^\dagger s_j = s_j$ hold true (see Remark A.1).

Given a number $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and a sequence $(s_j)_{j=0}^\kappa$ of complex $p \times q$ matrices, we consider, for all $m \in \mathbb{Z}_{0, \kappa}$, the triangular block Toeplitz matrices $\mathbf{S}_m^{(s)}$ and $\mathbb{S}_m^{(s)}$ defined by

$$\mathbf{S}_m^{(s)} := \begin{bmatrix} s_0 & 0 & 0 & \dots & 0 \\ s_1 & s_0 & 0 & \dots & 0 \\ s_2 & s_1 & s_0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_m & s_{m-1} & s_{m-2} & \dots & s_0 \end{bmatrix}, \quad \mathbb{S}_m^{(s)} := \begin{bmatrix} s_0 & s_1 & s_2 & \dots & s_m \\ 0 & s_0 & s_1 & \dots & s_{m-1} \\ 0 & 0 & s_0 & \dots & s_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & s_0 \end{bmatrix}, \quad (3.1)$$

respectively. Whenever it is clear which sequence is meant, we will write \mathbf{S}_m and \mathbb{S}_m instead of $\mathbf{S}_m^{(s)}$ and $\mathbb{S}_m^{(s)}$, respectively. Furthermore, for all $m \in \mathbb{Z}_{0, \kappa}$, we set

$$\mathbf{S}_m^\# := \mathbf{S}_m^{(r)} \quad \text{and} \quad \mathbb{S}_m^\# := \mathbb{S}_m^{(r)}, \quad (3.2)$$

where $(r_j)_{j=0}^\kappa$ denotes the reciprocal sequence corresponding to $(s_j)_{j=0}^\kappa$, i. e., $r_j := s_j^\#$ for all $j \in \mathbb{Z}_{0, \kappa}$. Now we introduce some block Hankel matrices. We will use the notations

$$H_n^\# := [s_{j+k}^\#]_{j,k=0}^n \quad (3.3)$$

for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$ and $K_n^\# := [s_{j+k+1}^\#]_{j,k=0}^n$ for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$. Furthermore, let

$$G_n^\# := [s_{j+k+2}^\#]_{j,k=0}^n \quad (3.4)$$

for all $n \in \mathbb{N}_0$ with $2n+2 \leq \kappa$ and let

$$y_{l,m}^\# := \text{col}(s_j^\#)_{j=l}^m \quad \text{and} \quad z_{l,m}^\# := [s_l^\#, s_{l+1}^\#, \dots, s_m^\#] \quad (3.5)$$

for all $l, m \in \mathbb{N}_0$ with $l \leq m \leq \kappa$. In the following, I_q stands for the identity matrix in $\mathbb{C}^{q \times q}$ and we will write $A \otimes B$ for the Kronecker product $A \otimes B := [a_{jk} B]_{j=1, \dots, p, k=1, \dots, q}$ of two complex matrices $A = [a_{jk}]_{j=1, \dots, p, k=1, \dots, q} \in \mathbb{C}^{p \times q}$ and $B \in \mathbb{C}^{r \times s}$. Observe that, for all $m \in \mathbb{N}$ and all $B \in \mathbb{C}^{r \times s}$, then $I_m \otimes B$ coincides with the block diagonal matrix $\text{diag}(B, \dots, B)$ with m diagonal blocks B .

Proposition 3.5. *Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^\kappa \in \mathcal{D}_{p \times q, \kappa}$. For all $m \in \mathbb{Z}_{0, \kappa}$, then $(s_j)_{j=0}^m$ belongs to $\mathcal{D}_{p \times q, m}$ and the equations*

$$\mathbf{S}_m^\dagger = \mathbf{S}_m^\#, \quad \mathbb{S}_m^\dagger = \mathbb{S}_m^\# \quad (3.6)$$

and

$$\mathbf{S}_m \mathbf{S}_m^\dagger = I_{m+1} \otimes (s_0 s_0^\dagger) = \mathbb{S}_m \mathbb{S}_m^\dagger, \quad \mathbf{S}_m^\dagger \mathbf{S}_m = I_{m+1} \otimes (s_0^\dagger s_0) = \mathbb{S}_m^\dagger \mathbb{S}_m$$

hold true. If, furthermore, $p = q$ and if $\mathcal{R}(s_0^*) = \mathcal{R}(s_0)$, then $\mathbf{S}_m \mathbf{S}_m^\dagger = \mathbf{S}_m^\dagger \mathbf{S}_m$ and $\mathbb{S}_m \mathbb{S}_m^\dagger = \mathbb{S}_m^\dagger \mathbb{S}_m$ for all $m \in \mathbb{Z}_{0, \kappa}$.

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Proof. Use [26, Remark 4.6, Theorem 4.21, and Lemma 3.6]. \square

The equations in (3.6) are characteristic for the class $\mathcal{D}_{p \times q, m}$, in some sense:

Proposition 3.6. *Let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m$ be a sequence of complex $p \times q$ matrices. Suppose that there is a sequence $(t_j)_{j=0}^m$ of complex $q \times p$ matrices such that $\mathbf{S}_m^\dagger = \mathbf{S}_m^{(t)}$ or $\mathbb{S}_m^\dagger = \mathbb{S}_m^{(t)}$ holds true. Then $(s_j)_{j=0}^m$ belongs to $\mathcal{D}_{p \times q, m}$.*

Proof. Use [26, Definition 2.2, Notation 2.5, and Propositions 2.18 and 4.4]. \square

In anticipation of later applications to Hankel non-negative definite sequences of matrices, we introduced in [25, Definition 4.16] a slight modification of first term domination.

Definition 3.7. Let $m \in \mathbb{N}$. A sequence $(s_j)_{j=0}^m$ of complex $p \times q$ matrices with $(s_j)_{j=0}^{m-1} \in \mathcal{D}_{p \times q, m-1}$ is called *nearly first term dominant*. The set of all nearly first term dominant sequences $(s_j)_{j=0}^m$ will be denoted by $\tilde{\mathcal{D}}_{p \times q, m}$. We also set $\tilde{\mathcal{D}}_{p \times q, 0} := \mathcal{D}_{p \times q, 0}$ and $\tilde{\mathcal{D}}_{p \times q, \infty} := \mathcal{D}_{p \times q, \infty}$.

Now we state some inclusions between the classes of sequences of complex $q \times q$ matrices introduced above.

Proposition 3.8. *Let $\alpha \in \mathbb{R}$ and let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$. Then:*

- (a) $\mathcal{K}_{q, \kappa, \alpha}^{\geq, e} \subseteq \mathcal{H}_{q, \kappa}^{\geq, e} \subseteq \mathcal{D}_{q \times q, \kappa} \subseteq \tilde{\mathcal{D}}_{q \times q, \kappa}$.
- (b) $\mathcal{K}_{q, 2\kappa, \alpha}^{\geq} \subseteq \mathcal{H}_{q, 2\kappa}^{\geq} \subseteq \tilde{\mathcal{D}}_{q \times q, 2\kappa}$.
- (c) $\mathcal{K}_{q, \kappa, \alpha}^{\geq, e} \subseteq \mathcal{K}_{q, \kappa, \alpha}^{\geq} \subseteq \tilde{\mathcal{D}}_{q \times q, \kappa}$.
- (d) $\mathcal{K}_{q, \kappa, \alpha}^> \cup \mathcal{K}_{q, \kappa, \alpha}^{\geq, \text{cd}} \subseteq \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$.

Proof. For all $n \in \mathbb{N}_0$ we have $\mathcal{K}_{q, 2n, \alpha}^{\geq, e} \subseteq \mathcal{H}_{q, 2n}^{\geq, e}$ by Lemma 2.4 and $\mathcal{K}_{q, 2n+1, \alpha}^{\geq, e} \subseteq \mathcal{H}_{q, 2n+1}^{\geq, e}$ by (2.1). In view of [25, Proposition 4.24] and the Definitions 3.7 and 3.3, hence $\mathcal{K}_{q, m, \alpha}^{\geq, e} \subseteq \mathcal{H}_{q, m}^{\geq, e} \subseteq \mathcal{D}_{q \times q, m} \subseteq \tilde{\mathcal{D}}_{q \times q, m}$ for all $m \in \mathbb{N}_0$. We have $\mathcal{K}_{q, 0, \alpha}^{\geq} \subseteq \mathcal{H}_{q, 0}^{\geq} \subseteq \tilde{\mathcal{D}}_{q \times q, 0}$ by the definition of $\mathcal{K}_{q, 0, \alpha}^{\geq}$, $\mathcal{H}_{q, 0}^{\geq}$, and $\tilde{\mathcal{D}}_{q \times q, 0}$ and, for all $n \in \mathbb{N}$, furthermore $\mathcal{K}_{q, 2n, \alpha}^{\geq} \subseteq \mathcal{H}_{q, 2n}^{\geq} \subseteq \tilde{\mathcal{D}}_{q \times q, 2n}$ by (1.4) and [25, Proposition 4.25]. Because of the definition of $\mathcal{K}_{q, \infty, \alpha}^{\geq}$ and $\mathcal{H}_{q, \infty}^{\geq}$, hence $\mathcal{K}_{q, \infty, \alpha}^{\geq} \subseteq \mathcal{H}_{q, \infty}^{\geq}$, which, in view of the definition of $\mathcal{K}_{q, \infty, \alpha}^{\geq, e}$, $\mathcal{H}_{q, \infty}^{\geq, e}$, and $\tilde{\mathcal{D}}_{q \times q, \infty}$ and [25, Proposition 4.24], implies $\mathcal{K}_{q, \infty, \alpha}^{\geq, e} \subseteq \mathcal{H}_{q, \infty}^{\geq, e} \subseteq \mathcal{D}_{q \times q, \infty} \subseteq \tilde{\mathcal{D}}_{q \times q, \infty}$ and $\mathcal{K}_{q, \infty, \alpha}^{\geq} \subseteq \mathcal{H}_{q, \infty}^{\geq} \subseteq \tilde{\mathcal{D}}_{q \times q, \infty}$. Thus, (a) and (b) are proved. From the definition of $\mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$ and Remark 2.1 we conclude $\mathcal{K}_{q, \kappa, \alpha}^{\geq, e} \subseteq \mathcal{K}_{q, \kappa, \alpha}^{\geq}$. Now let $n \in \mathbb{N}_0$ and $(s_j)_{j=0}^{2n+1} \in \mathcal{K}_{q, 2n+1, \alpha}^{\geq}$. Then we get $\{(s_j)_{j=0}^{2n}, (s_{\alpha \triangleright j})_{j=0}^{2n}\} \subseteq \mathcal{H}_{q, 2n}^{\geq} \subseteq \tilde{\mathcal{D}}_{q \times q, 2n}$ by (1.5) and (b). In the case $n = 0$, we have then $(s_j)_{j=0}^0 \in \mathcal{D}_{q \times q, 0}$, which, in view of Definition 3.7, implies $(s_j)_{j=0}^1 \in$

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$\tilde{\mathcal{D}}_{q \times q, 1}$. Now suppose $n \geq 1$. Because of (1.3) and Definition 3.7, we obtain then $\{(s_j)_{j=0}^{2n-1}, (-\alpha s_j + s_{j+1})_{j=0}^{2n-1}\} \subseteq \mathcal{D}_{q \times q, 2n-1}$, which, in view of Definition 3.3, implies

$$\begin{aligned} \mathcal{N}(s_0) &\subseteq \bigcap_{j=0}^{2n-1} \mathcal{N}(s_j) \subseteq \left[\bigcap_{j=0}^{2n-1} \mathcal{N}(s_j) \right] \cap \mathcal{N}(-\alpha s_0 + s_1) \\ &\subseteq \left[\bigcap_{j=0}^{2n-1} \mathcal{N}(s_j) \right] \cap \mathcal{N}(-\alpha s_{2n-1} + s_{2n}) \subseteq \bigcap_{j=0}^{2n} \mathcal{N}(s_j) \end{aligned}$$

and, analogously, $\bigcup_{j=0}^{2n} \mathcal{R}(s_j) \subseteq \mathcal{R}(s_0)$. Hence, in view of Definition 3.3, we have then $(s_j)_{j=0}^{2n} \in \mathcal{D}_{q \times q, 2n}$. Consequently, because of Definition 3.7, we get $(s_j)_{j=0}^{2n+1} \in \tilde{\mathcal{D}}_{q \times q, 2n+1}$. Taking additionally into account (b), we have thus proved (c). Furthermore, we know from [21, Propositions 2.20 and 5.9] that (d) holds true. \square

4. The $[+, \alpha]$ -transform of a matrix sequence

This section is of technical nature. We study a particular transform of finite or infinite sequences of $p \times q$ matrices. This transform plays an intermediate role in our construction of a Schur-Stieltjes-type algorithm for sequences of matrices.

Definition 4.1. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then we call the sequence $(s_j^{[+, \alpha]})_{j=0}^\kappa$ which is given, for all $j \in \mathbb{Z}_{0, \kappa}$, by $s_j^{[+, \alpha]} := -\alpha s_{j-1} + s_j$ where $s_{-1} := 0_{p \times q}$, the $[+, \alpha]$ -transform of $(s_j)_{j=0}^\kappa$.

Obviously, the $[+, \alpha]$ -transform of $(s_j)_{j=0}^\kappa$ is connected with the sequence $(s_{\alpha \triangleright j})_{j=0}^{\kappa-1}$ given in (1.3) via $s_{j+1}^{[+, \alpha]} = s_{\alpha \triangleright j}$ for all $j \in \mathbb{Z}_{0, \kappa-1}$. Furthermore, we have

$$s_0^{[+, \alpha]} = s_0. \quad (4.1)$$

Let $n \in \mathbb{N}_0$. Then, let

$$T_{q,n} := [\delta_{j,k+1} I_q]_{j,k=0}^n \quad (4.2)$$

where δ_{lm} is the Kronecker delta given by $\delta_{lm} := 1$ if $l = m$ and $\delta_{lm} := 0$ if $l \neq m$. Obviously, the function $R_{q,n}: \mathbb{C} \rightarrow \mathbb{C}^{(n+1)q \times (n+1)q}$ defined by

$$R_{q,n}(z) := (I_{(n+1)q} - zT_{q,n})^{-1}$$

admits the block representation

$$R_{q,n}(z) = \begin{bmatrix} I_q & 0 & 0 & \dots & 0 \\ zI_q & I_q & 0 & \dots & 0 \\ z^2 I_q & zI_q & I_q & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z^n I_q & z^{n-1} I_q & z^{n-2} I_q & \dots & I_q \end{bmatrix} \quad (4.3)$$

for all $z \in \mathbb{C}$.

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Remark 4.2. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices with $[+, \alpha]$ -transform $(t_j)_{j=0}^\kappa$. Then:

- (a) In view of Definition 4.1, for all $j \in \mathbb{Z}_{0,\kappa}$, the equation $s_j = \sum_{l=0}^j \alpha^{j-l} t_l$ holds true and, in particular, $\mathcal{R}(s_j) \subseteq \sum_{l=0}^j \mathcal{R}(t_l)$ and $\bigcap_{l=0}^j \mathcal{N}(t_l) \subseteq \mathcal{N}(s_j)$.
- (b) In view of (4.3), (3.1), and (a), for all $m \in \mathbb{Z}_{0,\kappa}$, the block Toeplitz matrices $\mathbf{S}_m^{[+, \alpha]} := \mathbf{S}_m^{(t)}$ and $\mathbb{S}_m^{[+, \alpha]} := \mathbb{S}_m^{(t)}$ admit the representations $\mathbf{S}_m^{[+, \alpha]} = \mathbf{S}_m[R_{q,m}(\alpha)]^{-1} = [R_{p,m}(\alpha)]^{-1} \mathbf{S}_m$ and $\mathbb{S}_m^{[+, \alpha]} = \mathbb{S}_m[R_{q,m}(\bar{\alpha})]^{-*} = [R_{p,m}(\bar{\alpha})]^{-*} \mathbb{S}_m$.
- (c) In view of (a), (4.1), and Definition 4.1, the inclusion $\bigcup_{j=0}^\kappa \mathcal{R}(t_j) \subseteq \mathcal{R}(t_0)$ holds true if and only if $\bigcup_{j=0}^\kappa \mathcal{R}(s_j) \subseteq \mathcal{R}(s_0)$, and, furthermore, $\mathcal{N}(t_0) \subseteq \bigcap_{j=0}^\kappa \mathcal{N}(t_j)$ if and only if $\mathcal{N}(s_0) \subseteq \bigcap_{j=0}^\kappa \mathcal{N}(s_j)$.

Remark 4.3. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices with $[+, \alpha]$ -transform $(t_j)_{j=0}^\kappa$. Then:

- (a) Definition 3.3 and Remark 4.2(c) show that $(t_j)_{j=0}^\kappa \in \mathcal{D}_{p \times q, \kappa}$ if and only if $(s_j)_{j=0}^\kappa \in \mathcal{D}_{p \times q, \kappa}$.
- (b) Definition 3.7 and (a) show that $(t_j)_{j=0}^\kappa \in \tilde{\mathcal{D}}_{p \times q, \kappa}$ if and only if $(s_j)_{j=0}^\kappa \in \tilde{\mathcal{D}}_{p \times q, \kappa}$.
- (c) By Definition 4.1, then $(t_j^*)_{j=0}^\kappa$ is exactly the $[+, \bar{\alpha}]$ -transform of $(s_j^*)_{j=0}^\kappa$.
- (d) If $p = q$, in view of (c) and Remark 4.2(a), then $s_j^* = s_j$ for all $j \in \mathbb{Z}_{0,\kappa}$ if and only if $(t_j^*)_{j=0}^\kappa$ is the $[+, \bar{\alpha}]$ -transform of $(s_j)_{j=0}^\kappa$.
- (e) If $p = q$ and $\alpha \in \mathbb{R}$, in view of (d), then $s_j^* = s_j$ for all $j \in \mathbb{Z}_{0,\kappa}$ if and only if $t_j^* = t_j$ for all $j \in \mathbb{Z}_{0,\kappa}$.

Remark 4.4. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $n \in \mathbb{N}$. For all $m \in \mathbb{Z}_{1,n}$, let $p_m, q_m \in \mathbb{N}$, let $(s_j^{(m)})_{j=0}^\kappa$ be a sequence of complex $p_m \times q_m$ matrices, let $(t_j^{(m)})_{j=0}^\kappa$ be its $[+, \alpha]$ -transform, let $L_m \in \mathbb{C}^{p \times p_m}$, and let $R_m \in \mathbb{C}^{q_m \times q}$. Then $(\sum_{m=1}^n L_m t_j^{(m)} R_m)_{j=0}^\kappa$ is exactly the $[+, \alpha]$ -transform of $(\sum_{m=1}^n L_m s_j^{(m)} R_m)_{j=0}^\kappa$.

Remark 4.5. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $n \in \mathbb{N}$. For all $m \in \mathbb{Z}_{1,n}$, let $p_m, q_m \in \mathbb{N}$ and let $(s_j^{(m)})_{j=0}^\kappa$ be a sequence of complex $p_m \times q_m$ matrices with $[+, \alpha]$ -transform $(t_j^{(m)})_{j=0}^\kappa$. Then $(\text{diag}[t_j^{(m)}]_{m=1}^n)_{j=0}^\kappa$ is exactly the $[+, \alpha]$ -transform of $(\text{diag}[s_j^{(m)}]_{m=1}^n)_{j=0}^\kappa$.

Let $\alpha \in \mathbb{C}$. In order to prepare the basic construction in Section 6, we study the reciprocal sequence corresponding to the $[+, \alpha]$ -transform of a sequence. Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices with $[+, \alpha]$ -transform $(u_j)_{j=0}^\kappa$. Then we define $(s_j^{[\sharp, \alpha]})_{j=0}^\kappa$ by

$$s_j^{[\sharp, \alpha]} := u_j^\sharp \quad (4.4)$$

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for all $j \in \mathbb{Z}_{0,\kappa}$, i.e., the sequence $(s_j^{[\sharp, \alpha]})_{j=0}^\kappa$ is defined to be the reciprocal sequence corresponding to the $[+, \alpha]$ -transform of $(s_j)_{j=0}^\kappa$. The following result describes a useful interrelation between the sequence introduced in (4.4) and the reciprocal sequence corresponding to the original sequence $(s_j)_{j=0}^\kappa$:

Lemma 4.6. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. For all $j \in \mathbb{Z}_{0,\kappa}$, then*

$$s_j^{[\sharp, \alpha]} = \sum_{l=0}^j \alpha^{j-l} s_l^\sharp. \quad (4.5)$$

Proof. For $j = 0$ equation (4.5) holds obviously. If $\kappa = 0$, then the proof is finished.

Let $\kappa \geq 1$. According to Definition 3.1, we have $s_j^\sharp = s_0^\dagger s_0 s_j^\sharp$ for all $j \in \mathbb{Z}_{0,\kappa}$ and hence, in view of (4.1), furthermore

$$s_1^{[\sharp, \alpha]} = -s_0^\dagger(-\alpha s_0 + s_1) s_0^\sharp = \alpha s_0^\sharp + s_1^\sharp = \sum_{l=0}^1 \alpha^{1-l} s_l^\sharp.$$

Thus, if $\kappa = 1$, then the proof is complete.

Now suppose $\kappa \geq 2$. Then the considerations above show that there is a number $k \in \mathbb{Z}_{1,\kappa-1}$ such that (4.5) holds true for all $j \in \mathbb{Z}_{0,k}$. Consequently,

$$\begin{aligned} s_{k+1}^{[\sharp, \alpha]} &= -\sum_{j=0}^k s_0^\dagger(-\alpha s_{k-j} + s_{k+1-j}) \sum_{l=0}^j \alpha^{j-l} s_l^\sharp \\ &= s_0^\dagger s_0 \sum_{l=0}^k \alpha^{k+1-l} s_l^\sharp + \sum_{j=0}^{k-1} s_0^\dagger s_{k-j} \sum_{l=0}^j \alpha^{j-l+1} s_l^\sharp - \sum_{j=0}^k s_0^\dagger s_{k+1-j} \sum_{l=0}^j \alpha^{j-l} s_l^\sharp \\ &= \sum_{l=0}^k \alpha^{k+1-l} s_l^\sharp + \sum_{j=1}^k s_0^\dagger s_{k+1-j} \sum_{l=0}^{j-1} \alpha^{j-l} s_l^\sharp - \sum_{j=0}^k s_0^\dagger s_{k+1-j} \sum_{l=0}^j \alpha^{j-l} s_l^\sharp \\ &= \sum_{l=0}^k \alpha^{k+1-l} s_l^\sharp - \sum_{j=0}^{(k+1)-1} s_0^\dagger s_{k+1-j} s_j^\sharp = \sum_{l=0}^{k+1} \alpha^{k+1-l} s_l^\sharp. \end{aligned}$$

Thus, the assertion follows inductively. \square

Remark 4.7. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Let the sequence $(r_j)_{j=0}^\kappa$ be given by $r_j := s_j^{[\sharp, \alpha]}$ for all $j \in \mathbb{Z}_{0,\kappa}$. Then we define, for all $n \in \mathbb{Z}_{0,\kappa}$, the block Toeplitz matrices $\mathbf{S}_n^{[\sharp, \alpha]} := \mathbf{S}_n^{(r)}$ and $\mathbb{S}_n^{[\sharp, \alpha]} := \mathbb{S}_n^{(r)}$. In view of (3.1), (4.3), and Lemma 4.6, one can easily see then that

$$\mathbf{S}_n^{[\sharp, \alpha]} = [R_{q,n}(\alpha)] \mathbf{S}_n^\sharp = \mathbf{S}_n^\sharp [R_{p,n}(\alpha)] \quad \text{and} \quad \mathbb{S}_n^{[\sharp, \alpha]} = [R_{q,n}(\bar{\alpha})]^* \mathbb{S}_n^\sharp = \mathbb{S}_n^\sharp [R_{p,n}(\bar{\alpha})]^*$$

hold true for all $n \in \mathbb{Z}_{0,\kappa}$. If $(s_j)_{j=0}^\kappa \in \mathcal{D}_{p \times q, \kappa}$, then from Proposition 3.5 we see that, for all $n \in \mathbb{Z}_{0,\kappa}$ these equations admit the reformulation

$$\mathbf{S}_n^{[\sharp, \alpha]} = [R_{q,n}(\alpha)] \mathbf{S}_n^\dagger = \mathbf{S}_n^\dagger [R_{p,n}(\alpha)] \quad \text{and} \quad \mathbb{S}_n^{[\sharp, \alpha]} = [R_{q,n}(\bar{\alpha})]^* \mathbb{S}_n^\dagger = \mathbb{S}_n^\dagger [R_{p,n}(\bar{\alpha})]^*.$$

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Lemma 4.8. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then $[\mathcal{R}(s_0)]^\perp \subseteq \bigcap_{j=0}^\kappa \mathcal{N}(s_j^{[\sharp, \alpha]})$ and $\bigcup_{j=0}^\kappa \mathcal{R}(s_j^{[\sharp, \alpha]}) \subseteq [\mathcal{N}(s_0)]^\perp$. In particular, $(s_j^{[\sharp, \alpha]})_{j=0}^\kappa$ belongs to $\mathcal{D}_{q \times p, \kappa}$.*

Proof. Remark A.1(a) shows that

$$[\mathcal{R}(s_0)]^\perp = \mathcal{N}(s_0^*) = \mathcal{N}(s_0^\dagger) \quad \text{and} \quad [\mathcal{N}(s_0)]^\perp = \mathcal{R}(s_0^*) = \mathcal{R}(s_0^\dagger). \quad (4.6)$$

Thus, from Definition 3.1 we see that $[\mathcal{R}(s_0)]^\perp \subseteq \mathcal{N}(s_k^\sharp)$ for all $k \in \mathbb{Z}_{0, \kappa}$. Taking Lemma 4.6 into account, we then get $[\mathcal{R}(s_0)]^\perp \subseteq \bigcap_{j=0}^\kappa \mathcal{N}(s_j^{[\sharp, \alpha]})$. Furthermore, from Lemma 4.6, Definition 3.1, and (4.6) we see that $\bigcup_{j=0}^\kappa \mathcal{R}(s_j^{[\sharp, \alpha]}) \subseteq \bigcup_{j=0}^\kappa \mathcal{R}(s_j^\sharp) \subseteq \mathcal{R}(s_0^\dagger) = [\mathcal{N}(s_0)]^\perp$. Taking into account that $s_0^{[\sharp, \alpha]} = (s_0^{[+, \alpha]})^\dagger = s_0^\dagger$ holds true, the proof is complete. \square

Now we study the $[+, \alpha]$ -transform of the sequence introduced in (4.4).

Lemma 4.9. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then the $[+, \alpha]$ -transform of $(s_j^{[\sharp, \alpha]})_{j=0}^\kappa$ is exactly the reciprocal sequence $(s_j^\sharp)_{j=0}^\kappa$ corresponding to $(s_j)_{j=0}^\kappa$.*

Proof. Let the sequence $(r_j)_{j=0}^\kappa$ be given by $r_j := s_j^{[\sharp, \alpha]}$ for all $j \in \mathbb{Z}_{0, \kappa}$. From Lemma 4.6 we know then that $r_j = \sum_{l=0}^j \alpha^{j-l} s_l^\sharp$ holds true for all $j \in \mathbb{Z}_{0, \kappa}$. Using (4.1) and Definition 4.1, we obtain then $r_0^{[+, \alpha]} = s_0^\sharp$ and, in the case $\kappa \geq 1$, for all $j \in \mathbb{Z}_{1, \kappa}$, furthermore

$$r_j^{[+, \alpha]} = -\alpha r_{j-1} + r_j = -\alpha \sum_{l=0}^{j-1} \alpha^{(j-1)-l} s_l^\sharp + \sum_{l=0}^j \alpha^{j-l} s_l^\sharp = s_j^\sharp. \quad \square$$

Lemma 4.10. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ and $(t_j)_{j=0}^\kappa$ be sequences of complex $p \times q$ matrices. Then $s_j^{[\sharp, \alpha]} = t_j^{[\sharp, \alpha]}$ for all $j \in \mathbb{Z}_{0, \kappa}$ if and only if $s_0 s_0^\dagger s_j s_0^\dagger s_0 = t_0 t_0^\dagger t_j t_0^\dagger t_0$ for all $j \in \mathbb{Z}_{0, \kappa}$.*

Proof. According to Lemmas 4.9 and 4.6, the equation $s_j^{[\sharp, \alpha]} = t_j^{[\sharp, \alpha]}$ holds for all $j \in \mathbb{Z}_{0, \kappa}$ if and only if $s_j^\sharp = t_j^\sharp$ for all $j \in \mathbb{Z}_{0, \kappa}$. The application of [26, Proposition 5.11] completes the proof. \square

We finish this section with some considerations on the arithmetics of the $[+, \alpha]$ -transform.

Lemma 4.11. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices.*

- (a) *If $\gamma \in \mathbb{C}$, then $((\gamma s_j)^{[\sharp, \alpha]})_{j=0}^\kappa = (\gamma^\dagger s_j^{[\sharp, \alpha]})_{j=0}^\kappa$ and $((\gamma^j s_j)^{[\sharp, \alpha]})_{j=0}^\kappa = (\gamma^j s_j^{[\sharp, \alpha]})_{j=0}^\kappa$.*
- (b) *If $m \in \mathbb{N}$ and $L \in \mathbb{C}^{m \times p}$ with $\mathcal{R}(L^*) = \mathcal{R}(s_0)$, then $((L s_j)^{[\sharp, \alpha]})_{j=0}^\kappa = (s_j^{[\sharp, \alpha]} L^\dagger)_{j=0}^\kappa$.*

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- (c) If $n \in \mathbb{N}$ and $R \in \mathbb{C}^{q \times n}$ with $\mathcal{R}(s_0^*) = \mathcal{R}(R)$, then $((s_j R)^{[\sharp, \alpha]})_{j=0}^\kappa = (R^\dagger s_j^{[\sharp, \alpha]})_{j=0}^\kappa$.
- (d) If $m, n \in \mathbb{N}$, $L \in \mathbb{C}^{m \times p}$ with $\mathcal{R}(L^*) = \mathcal{R}(s_0)$, and $R \in \mathbb{C}^{q \times n}$ with $\mathcal{R}(s_0^*) = \mathcal{R}(R)$, then $((L s_j R)^{[\sharp, \alpha]})_{j=0}^\kappa = (R^\dagger s_j^{[\sharp, \alpha]} L^\dagger)_{j=0}^\kappa$.
- (e) If $m, n \in \mathbb{N}$, $U \in \mathbb{C}^{m \times p}$ with $U^* U = I_p$, and $V \in \mathbb{C}^{q \times n}$ with $V V^* = I_q$, then $((U s_j V)^{[\sharp, \alpha]})_{j=0}^\kappa = (V^* s_j^{[\sharp, \alpha]} U^*)_{j=0}^\kappa$.
- (f) $(s_j^{[\sharp, \alpha]})^* = t_j^{[\sharp, \bar{\alpha}]}$ for all $j \in \mathbb{Z}_{0, \kappa}$ where the sequence $(t_j)_{j=0}^\kappa$ is given by $t_j := s_j^*$ for all $j \in \mathbb{Z}_{0, \kappa}$.

Proof. Part (a) follows from Lemma 4.6 and [26, Remarks 5.8 and 5.9]. Parts (b), (c), and (d) are immediate consequences of Lemma 4.6 and [26, Lemma 5.18]. In view of Lemma 4.6 and [26, Lemma 5.19], we see that (e) holds true. Finally, using (4.4), [26, Proposition 5.16], and Remark 4.3(c), we obtain (f). \square

Remark 4.12. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $n \in \mathbb{N}$. For all $m \in \mathbb{Z}_{1, n}$, let $p_m, q_m \in \mathbb{N}$ and let $(s_j^{(m)})_{j=0}^\kappa$ be a sequence of complex $p_m \times q_m$ matrices. Then from Remark 3.2, Lemma 4.6, and [26, Remark 5.20] we easily see that $(\text{diag}[(s_j^{(m)})^{[\sharp, \alpha]}]_{m=1}^n)_{j=0}^\kappa = ((\text{diag}[s_j^{(m)}]_{m=1}^n)^{[\sharp, \alpha]})_{j=0}^\kappa$.

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Similar as in [25, Section 6], the main theme of this section is the investigation of the interplay between various block Hankel matrices. In order to describe a typical situation, let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n}$ be a sequence from $\mathbb{C}^{q \times q}$. Denote by $(u_j)_{j=0}^{2n}$ the $[+, \alpha]$ -transform of $(s_j)_{j=0}^{2n}$ and by $(u_j^\sharp)_{j=0}^{2n}$ the reciprocal sequence corresponding to $(u_j)_{j=0}^{2n}$. Then we are going to find formulas which connect various block Hankel matrices built from the sequences $(u_j)_{j=0}^{2n}$ and $(u_j^\sharp)_{j=0}^{2n}$. This section should be compared with [25, Section 6], where formulas connecting several block Hankel matrices formed from the sequence $(s_j)_{j=0}^{2n}$ and its corresponding reciprocal sequence $(s_j^\sharp)_{j=0}^{2n}$ were derived. Our strategy of proof is essentially based on using results from [25, Section 6].

Let $v_{q,0} := I_q$ and, for all $n \in \mathbb{N}$, let

$$v_{q,n} := \begin{bmatrix} I_q \\ 0_{nq \times q} \end{bmatrix}, \quad \Delta_{q,n} := \begin{bmatrix} I_{nq} \\ 0_{q \times nq} \end{bmatrix}, \quad \text{and} \quad \nabla_{q,n} := \begin{bmatrix} 0_{q \times nq} \\ I_{nq} \end{bmatrix}. \quad (5.1)$$

Lemma 5.1. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices with $[+, \alpha]$ -transform $(t_j)_{j=0}^\kappa$.*

- (a) *Let $n \in \mathbb{N}$ with $2n \leq \kappa$. Then the block Hankel matrix $H_n^{[+, \alpha]} := H_n^{(t)}$ admits the representations*

$$\begin{aligned} H_n^{[+, \alpha]} &= [R_{p,n}(\alpha)]^{-1} H_n [R_{q,n}(\bar{\alpha})]^{-*} + \alpha \nabla_{p,n} (-\alpha H_{n-1} + K_{n-1}) \nabla_{q,n}^* \\ &= [R_{p,n}(\alpha)]^{-1} H_n [R_{q,n}(\bar{\alpha})]^{-*} + \alpha \nabla_{p,n} H_{\alpha \triangleright n-1} \nabla_{q,n}^*. \end{aligned} \quad (5.2)$$

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- (b) Let $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$. Then the block Hankel matrix $K_n^{[+, \alpha]} := K_n^{(t)}$ admits the representation $K_n^{[+, \alpha]} = -\alpha H_n + K_n$, i. e., $K_n^{[+, \alpha]} = H_{\alpha \triangleright n}$ holds true.
- (c) Let $n \in \mathbb{N}_0$ with $2n + 2 \leq \kappa$. Then the block Hankel matrix $G_n^{[+, \alpha]} := G_n^{(t)}$ admits the representation $G_n^{[+, \alpha]} = -\alpha K_n + G_n$, i. e., $G_n^{[+, \alpha]} = K_{\alpha \triangleright n}$ holds true.
- (d) Let $n \in \mathbb{N}$ with $2n \leq \kappa$. Then the matrix $\mathbb{L}_n^{[+, \alpha]} := \mathbb{L}_n^{(t)}$ admits the representation $\mathbb{L}_n^{[+, \alpha]} = K_{\alpha \triangleright n-1} - y_{\alpha \triangleright 0, n-1} s_0^\dagger z_{\alpha \triangleright 0, n-1}$.

Proof. Using the block representation $H_n = \begin{bmatrix} s_0 & z_{1,n} \\ y_{1,n} & G_{n-1} \end{bmatrix}$ of H_n , one can easily see that

$$\begin{aligned} H_n^{[+, \alpha]} &= \begin{bmatrix} s_0 & -\alpha z_{0, n-1} + z_{1, n} \\ -\alpha y_{0, n-1} + y_{1, n} & -\alpha K_{n-1} + G_{n-1} \end{bmatrix} \\ &= -\alpha \begin{bmatrix} 0_{q \times q} & 0_{q \times nq} \\ y_{0, n-1} & K_{n-1} \end{bmatrix} - \alpha \begin{bmatrix} 0_{q \times q} & z_{0, n-1} \\ 0_{nq \times q} & K_{n-1} \end{bmatrix} + \alpha \begin{bmatrix} 0_{q \times q} & 0_{q \times nq} \\ 0_{nq \times q} & K_{n-1} \end{bmatrix} + H_n \\ &= -\alpha T_{p,n} H_n - \alpha H_n T_{q,n}^* + \alpha \nabla_{p,n} K_{n-1} \nabla_{q,n}^* + H_n. \end{aligned} \quad (5.3)$$

Taking into account

$$\begin{aligned} [R_{p,n}(\alpha)]^{-1} H_n [R_{q,n}(\overline{\alpha})]^{-*} &= (I_{(n+1)p} - \alpha T_{p,n}) H_n (I_{(n+1)q} - \overline{\alpha} T_{q,n})^* \\ &= H_n - \alpha T_{p,n} H_n - \alpha H_n T_{q,n}^* + \alpha^2 T_{p,n} H_n T_{q,n}^* \end{aligned}$$

and $T_{p,n} H_n T_{q,n}^* = \nabla_{p,n} H_{n-1} \nabla_{q,n}^*$, from (5.3) we then conclude that (5.2) holds true. Thus, (a) is verified. The proof of the parts (b), (c), and (d) is straightforward. \square

In order to state interesting identities for block Hankel matrices, it seems to be useful to introduce some further notation. Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. For all $n \in \mathbb{N}_0$ and all $m \in \mathbb{Z}_{0,\kappa}$, we then set

$$\Xi_{n,m}^{(s)} := \begin{cases} s_m - s_0 s_0^\dagger s_m s_0^\dagger s_0 & \text{if } n = 0 \\ \text{diag}[0_{np \times nq}, s_m - s_0 s_0^\dagger s_m s_0^\dagger s_0] & \text{if } n \geq 1 \end{cases} \quad (5.4)$$

and write $\Xi_{n,m}$ instead of $\Xi_{n,m}^{(s)}$ if it is clear which sequence $(s_j)_{j=0}^\kappa$ of complex matrices is meant.

Remark 5.2. Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices, let $n \in \mathbb{N}_0$, and let $m \in \mathbb{Z}_{0,\kappa}$ be such that $\mathcal{N}(s_0) \subseteq \mathcal{N}(s_m)$ and $\mathcal{R}(s_m) \subseteq \mathcal{R}(s_0)$. In view of parts (b) and (c) of Remark A.1, then $s_m s_0^\dagger s_0 = s_m$ and $s_0 s_0^\dagger s_m = s_m$. Consequently, $\Xi_{n,m} = 0_{(n+1)p \times (n+1)q}$.

Remark 5.3. Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^\kappa \in \mathcal{D}_{p \times q, \kappa}$. In view of Definition 3.3 and Remark 5.2, then $\Xi_{n,m} = 0_{(n+1)p \times (n+1)q}$ for all $n \in \mathbb{N}_0$ and all $m \in \mathbb{Z}_{0,\kappa}$.

Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Furthermore, let the sequence $(s_j^{[\sharp, \alpha]})_{j=0}^\kappa$ be defined via (4.4). Then let $H_n^{[\sharp, \alpha]} := [s_{j+k}^{[\sharp, \alpha]}]_{j,k=0}^n$ for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, let

$$K_n^{[\sharp, \alpha]} := [s_{j+k+1}^{[\sharp, \alpha]}]_{j,k=0}^n \quad (5.5)$$

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for all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, and let

$$G_n^{[\sharp, \alpha]} := [s_{j+k+2}^{[\sharp, \alpha]}]_{j,k=0}^n \quad (5.6)$$

for all $n \in \mathbb{N}_0$ with $2n + 2 \leq \kappa$. Further, for all $l, m \in \mathbb{N}_0$ with $l \leq m \leq \kappa$, let $y_{l,m}^{[\sharp, \alpha]} := \text{col}(s_j^{[\sharp, \alpha]})_{j=l}^m$ and $z_{l,m}^{[\sharp, \alpha]} := [s_l^{[\sharp, \alpha]}, s_{l+1}^{[\sharp, \alpha]}, \dots, s_m^{[\sharp, \alpha]}]$ and let $y_{l,m}^{[+, \alpha]} := y_{l,m}^{(t)}$ and $z_{l,m}^{[+, \alpha]} := z_{l,m}^{(t)}$ where $(t_j)_{j=0}^\kappa$ denotes the $[+, \alpha]$ -transform of $(s_j)_{j=0}^\kappa$.

Proposition 5.4. *Let $\alpha \in \mathbb{C}$, let $n \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{2n} \in \tilde{\mathcal{D}}_{p \times q, 2n}$. Then*

$$H_n^{[\sharp, \alpha]} + \mathbf{S}_n^{[\sharp, \alpha]} H_n^{[+, \alpha]} \mathbb{S}_n^{[\sharp, \alpha]} = y_{0,n}^{[\sharp, \alpha]} v_{p,n}^* + v_{q,n} z_{0,n}^{[\sharp, \alpha]} \quad (5.7)$$

and

$$H_n^{[+, \alpha]} + \mathbf{S}_n^{[+, \alpha]} H_n^{[\sharp, \alpha]} \mathbb{S}_n^{[+, \alpha]} = y_{0,n}^{[+, \alpha]} v_{q,n}^* + v_{p,n} z_{0,n}^{[+, \alpha]} + \Xi_{n, 2n}. \quad (5.8)$$

Furthermore, if $n \geq 1$, then

$$\begin{aligned} H_n^{[\sharp, \alpha]} + \mathbf{S}_n^\dagger (H_n + \alpha R_{p,n}(\alpha) \nabla_{p,n} H_{\alpha \triangleright n-1} [R_{q,n}(\bar{\alpha}) \nabla_{q,n}]^*) \mathbb{S}_n^\dagger \\ = R_{q,n}(\alpha) y_{0,n}^\sharp v_{p,n}^* + v_{q,n} z_{0,n}^\sharp [R_{p,n}(\bar{\alpha})]^* \end{aligned} \quad (5.9)$$

and

$$\begin{aligned} H_n + \mathbf{S}_n H_n^{[\sharp, \alpha]} \mathbb{S}_n + \alpha R_{p,n}(\alpha) \nabla_{p,n} H_{\alpha \triangleright n-1} [R_{q,n}(\bar{\alpha}) \nabla_{q,n}]^* \\ = y_{0,n} [R_{q,n}(\bar{\alpha}) v_{q,n}]^* + R_{p,n}(\alpha) v_{p,n} z_{0,n} + \Xi_{n, 2n}. \end{aligned} \quad (5.10)$$

Proof. Because of $(s_j)_{j=0}^{2n} \in \tilde{\mathcal{D}}_{p \times q, 2n}$ and Remark 3.4, we have

$$\begin{aligned} s_{2n}^{[+, \alpha]} - s_0^{[+, \alpha]} (s_0^{[+, \alpha]})^\dagger s_{2n}^{[+, \alpha]} (s_0^{[+, \alpha]})^\dagger s_0^{[+, \alpha]} \\ = -\alpha s_{2n-1} + s_{2n} + \alpha s_0 s_0^\dagger s_{2n-1} s_0^\dagger s_0 - s_0 s_0^\dagger s_{2n} s_0^\dagger s_0 = s_{2n} - s_0 s_0^\dagger s_{2n} s_0^\dagger s_0 \end{aligned}$$

and, according to Remark 4.3(b), the sequence $(s_j^{[+, \alpha]})_{j=0}^{2n}$ belongs to $\tilde{\mathcal{D}}_{p \times q, 2n}$. Thus, (5.7) and (5.8) immediately follow from [25, Theorem 6.1].

Now assume $n \geq 1$. Then $(s_j)_{j=0}^n$ belongs to $\mathcal{D}_{p \times q, n}$ and, thus, Proposition 3.5 yields $\mathbf{S}_n^\dagger = \mathbf{S}_n^\sharp$ and $\mathbb{S}_n^\dagger = \mathbb{S}_n^\sharp$. Consequently, from these equations, from Remark 4.7, and from Lemma 5.1(a) we get

$$\begin{aligned} \mathbf{S}_n^{[\sharp, \alpha]} H_n^{[+, \alpha]} \mathbb{S}_n^{[\sharp, \alpha]} \\ = \mathbf{S}_n^\sharp R_{p,n}(\alpha) \left([R_{p,n}(\alpha)]^{-1} H_n [R_{q,n}(\bar{\alpha})]^{-*} + \alpha \nabla_{p,n} H_{\alpha \triangleright n-1} \nabla_{q,n}^* \right) [R_{q,n}(\bar{\alpha})]^* \mathbb{S}_n^\sharp \\ = \mathbf{S}_n^\dagger (H_n + \alpha R_{p,n}(\alpha) \nabla_{p,n} H_{\alpha \triangleright n-1} [R_{q,n}(\bar{\alpha}) \nabla_{q,n}]^*) \mathbb{S}_n^\dagger. \end{aligned} \quad (5.11)$$

Using (5.1) and Remark 4.7, we also see

$$y_{0,n}^{[\sharp, \alpha]} = \mathbf{S}_n^{[\sharp, \alpha]} v_{p,n} = R_{q,n}(\alpha) \mathbf{S}_n^\sharp v_{p,n} = R_{q,n}(\alpha) y_{0,n}^\sharp \quad (5.12)$$

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and, similarly, $z_{0,n}^{[\sharp,\alpha]} = z_{0,n}^\sharp [R_{p,n}(\bar{\alpha})]^*$. Thus, (5.7), (5.11), and (5.12) imply (5.9). From Lemma 5.1(a) we know that (5.2) holds true. Remark 4.2(b) shows

$$\mathbf{S}_n^{[+,\alpha]} H_n^{[\sharp,\alpha]} \mathbb{S}_n^{[+,\alpha]} = [R_{p,n}(\alpha)]^{-1} \mathbf{S}_n H_n^{[\sharp,\alpha]} \mathbb{S}_n [R_{q,n}(\bar{\alpha})]^{-*}. \quad (5.13)$$

Because of Remark 4.2(b), we have

$$y_{0,n}^{[+,\alpha]} = \mathbf{S}_n^{[+,\alpha]} v_{q,n} = [R_{p,n}(\alpha)]^{-1} \mathbf{S}_n v_{q,n} = [R_{p,n}(\alpha)]^{-1} y_{0,n} \quad (5.14)$$

and, similarly,

$$z_{0,n}^{[+,\alpha]} = z_{0,n} [R_{q,n}(\bar{\alpha})]^{-*}. \quad (5.15)$$

It is readily checked that $[R_{p,n}(\alpha)] \Xi_{n,2n} [R_{q,n}(\bar{\alpha})]^* = \Xi_{n,2n}$. Thus, multiplying equation (5.8) from the left by $R_{p,n}(\alpha)$ and from the right by $[R_{q,n}(\bar{\alpha})]^*$, and using (5.2), (5.13), (5.14), and (5.15), we see that (5.10) holds true. \square

Proposition 5.5. *Let $\alpha \in \mathbb{C}$, let $n \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{2n} \in \tilde{\mathcal{D}}_{p \times q, 2n}$. Then:*

(a)

$$H_n^{[\sharp,\alpha]} = -\mathbf{S}_n^{[\sharp,\alpha]} \left[H_n^{[+,\alpha]} - (y_{0,n}^{[+,\alpha]} v_{q,n}^* + v_{p,n} z_{0,n}^{[+,\alpha]}) \right] \mathbb{S}_n^{[\sharp,\alpha]} \quad (5.16)$$

and

$$H_n^{[+,\alpha]} = \Xi_{n,2n} - \mathbf{S}_n^{[+,\alpha]} \left[H_n^{[\sharp,\alpha]} - (y_{0,n}^{[\sharp,\alpha]} v_{q,n}^* + v_{p,n} z_{0,n}^{[\sharp,\alpha]}) \right] \mathbb{S}_n^{[+,\alpha]}. \quad (5.17)$$

(b) If $n \geq 1$, then

$$\begin{aligned} H_n^{[\sharp,\alpha]} &= -\mathbf{S}_n^\dagger [H_n + \alpha R_{p,n}(\alpha) \nabla_{p,n} H_{\alpha \triangleright n-1} [R_{q,n}(\bar{\alpha}) \nabla_{q,n}]^* \\ &\quad - (y_{0,n} [R_{q,n}(\bar{\alpha}) v_{q,n}]^* + R_{p,n}(\alpha) v_{p,n} z_{0,n})] \mathbb{S}_n^\dagger \end{aligned} \quad (5.18)$$

and

$$\begin{aligned} &H_n + \alpha R_{p,n}(\alpha) \nabla_{p,n} H_{\alpha \triangleright n-1} [R_{q,n}(\bar{\alpha}) \nabla_{q,n}]^* \\ &= \Xi_{n,2n} - \mathbf{S}_n \left[H_n^{[\sharp,\alpha]} - \left(R_{q,n}(\alpha) y_{0,n}^\sharp v_{p,n}^* + v_{q,n} z_{0,n}^\sharp [R_{p,n}(\bar{\alpha})]^* \right) \right] \mathbb{S}_n. \end{aligned} \quad (5.19)$$

(c) $\text{rank}(H_n^{[\sharp,\alpha]}) = \text{rank}[H_n^{[+,\alpha]} - (y_{0,n}^{[+,\alpha]} v_{q,n}^* + v_{p,n} z_{0,n}^{[+,\alpha]}) - \Xi_{n,2n}]$.

(d) If $p = q$, then

$$\det(H_n^{[\sharp,\alpha]}) = \left[(\det s_0)^\dagger \right]^{2n+2} \det \left(- \left[H_n^{[+,\alpha]} - (y_{0,n}^{[+,\alpha]} v_{q,n}^* + v_{p,n} z_{0,n}^{[+,\alpha]}) \right] \right).$$

Proof. (a) Since $(s_j)_{j=0}^{2n}$ belongs to $\tilde{\mathcal{D}}_{p \times q, 2n}$, Proposition 5.4 shows that (5.7) and (5.8) hold true. Furthermore, Remark 4.3(b) yields $(s_j^{[+,\alpha]})_{j=0}^{2n} \in \tilde{\mathcal{D}}_{p \times q, 2n}$ and, consequently, $(s_j^{[+,\alpha]})_{j=0}^n \in \mathcal{D}_{p \times q, n}$. From Proposition 3.5 we see then

$$\mathbf{S}_n^{[\sharp,\alpha]} = (\mathbf{S}_n^{[+,\alpha]})^\dagger \quad \text{and} \quad \mathbb{S}_n^{[\sharp,\alpha]} = (\mathbb{S}_n^{[+,\alpha]})^\dagger. \quad (5.20)$$

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Thus, Proposition 3.5 yields

$$\mathbf{S}_n^{[\sharp, \alpha]} \mathbf{S}_n^{[+, \alpha]} = I_{n+1} \otimes (s_0^{[\sharp, \alpha]} s_0^{[+, \alpha]}) \quad \text{and} \quad \mathbb{S}_n^{[+, \alpha]} \mathbb{S}_n^{[\sharp, \alpha]} = I_{n+1} \otimes (s_0^{[+, \alpha]} s_0^{[\sharp, \alpha]}).$$

This implies

$$\mathbf{S}_n^{[\sharp, \alpha]} \mathbf{S}_n^{[+, \alpha]} v_{q,n} = v_{q,n} s_0^{[\sharp, \alpha]} s_0^{[+, \alpha]} \quad \text{and} \quad v_{p,n}^* \mathbb{S}_n^{[+, \alpha]} \mathbb{S}_n^{[\sharp, \alpha]} = s_0^{[+, \alpha]} s_0^{[\sharp, \alpha]} v_{p,n}^*.$$

Consequently, we conclude

$$\mathbf{S}_n^{[\sharp, \alpha]} y_{0,n}^{[+, \alpha]} = \mathbf{S}_n^{[\sharp, \alpha]} \mathbf{S}_n^{[+, \alpha]} v_{q,n} = v_{q,n} s_0^{[\sharp, \alpha]} s_0^{[+, \alpha]} \quad (5.21)$$

and

$$z_{0,n}^{[+, \alpha]} \mathbb{S}_n^{[\sharp, \alpha]} = v_{p,n}^* \mathbb{S}_n^{[+, \alpha]} \mathbb{S}_n^{[\sharp, \alpha]} = s_0^{[+, \alpha]} s_0^{[\sharp, \alpha]} v_{p,n}^*. \quad (5.22)$$

From [25, Proposition 4.9(b)] we see that

$$s_0^{[\sharp, \alpha]} s_0^{[+, \alpha]} z_{0,n}^{[\sharp, \alpha]} = z_{0,n}^{[\sharp, \alpha]} \quad \text{and} \quad y_{0,n}^{[\sharp, \alpha]} s_0^{[+, \alpha]} s_0^{[\sharp, \alpha]} = y_{0,n}^{[\sharp, \alpha]}. \quad (5.23)$$

Using (5.21), (5.22), and (5.23), we get then

$$\begin{aligned} \mathbf{S}_n^{[\sharp, \alpha]} \left[y_{0,n}^{[+, \alpha]} v_{q,n}^* + v_{p,n} z_{0,n}^{[+, \alpha]} \right] \mathbb{S}_n^{[\sharp, \alpha]} &= \mathbf{S}_n^{[\sharp, \alpha]} y_{0,n}^{[+, \alpha]} z_{0,n}^{[\sharp, \alpha]} + y_{0,n}^{[\sharp, \alpha]} z_{0,n}^{[+, \alpha]} \mathbb{S}_n^{[\sharp, \alpha]} \\ &= v_{q,n} s_0^{[\sharp, \alpha]} s_0^{[+, \alpha]} z_{0,n}^{[\sharp, \alpha]} + y_{0,n}^{[\sharp, \alpha]} s_0^{[+, \alpha]} s_0^{[\sharp, \alpha]} v_{p,n}^* = v_{q,n} z_{0,n}^{[\sharp, \alpha]} + y_{0,n}^{[\sharp, \alpha]} v_{p,n}^*. \end{aligned} \quad (5.24)$$

Applying (5.24) to equation (5.7), we then obtain (5.16). Because of (5.20) and Proposition 3.5, we have

$$\mathbf{S}_n^{[+, \alpha]} \mathbf{S}_n^{[\sharp, \alpha]} = I_{n+1} \otimes (s_0^{[+, \alpha]} s_0^{[\sharp, \alpha]}) \quad \text{and} \quad \mathbb{S}_n^{[\sharp, \alpha]} \mathbb{S}_n^{[+, \alpha]} = I_{n+1} \otimes (s_0^{[\sharp, \alpha]} s_0^{[+, \alpha]}).$$

This provides us

$$\mathbf{S}_n^{[+, \alpha]} y_{0,n}^{[\sharp, \alpha]} = \mathbf{S}_n^{[+, \alpha]} \mathbf{S}_n^{[\sharp, \alpha]} v_{p,n} = v_{p,n} s_0^{[+, \alpha]} s_0^{[\sharp, \alpha]} \quad (5.25)$$

and

$$z_{0,n}^{[\sharp, \alpha]} \mathbb{S}_n^{[+, \alpha]} = v_{q,n}^* \mathbb{S}_n^{[\sharp, \alpha]} \mathbb{S}_n^{[+, \alpha]} = s_0^{[\sharp, \alpha]} s_0^{[+, \alpha]} v_{q,n}^*. \quad (5.26)$$

Since $(s_j^{[+, \alpha]})_{j=0}^{2n}$ belongs to $\tilde{\mathcal{D}}_{p \times q, 2n}$, from Remark A.1 we see that the equations

$$y_{0,n}^{[+, \alpha]} s_0^{[\sharp, \alpha]} s_0^{[+, \alpha]} = y_{0,n}^{[+, \alpha]} \quad \text{and} \quad s_0^{[+, \alpha]} s_0^{[\sharp, \alpha]} z_{0,n}^{[+, \alpha]} = z_{0,n}^{[+, \alpha]} \quad (5.27)$$

hold true. Using (5.25), (5.26), and (5.27), we infer

$$\begin{aligned} \mathbf{S}_n^{[+, \alpha]} (y_{0,n}^{[\sharp, \alpha]} v_{p,n}^* + v_{q,n} z_{0,n}^{[\sharp, \alpha]}) \mathbb{S}_n^{[+, \alpha]} &= \mathbf{S}_n^{[+, \alpha]} y_{0,n}^{[\sharp, \alpha]} z_{0,n}^{[+, \alpha]} + y_{0,n}^{[+, \alpha]} z_{0,n}^{[\sharp, \alpha]} \mathbb{S}_n^{[+, \alpha]} \\ &= v_{p,n} s_0^{[+, \alpha]} s_0^{[\sharp, \alpha]} z_{0,n}^{[+, \alpha]} + y_{0,n}^{[+, \alpha]} s_0^{[\sharp, \alpha]} s_0^{[+, \alpha]} v_{q,n}^* = v_{p,n} z_{0,n}^{[+, \alpha]} + y_{0,n}^{[+, \alpha]} v_{q,n}^*. \end{aligned} \quad (5.28)$$

From (5.8) and (5.28) we then get (5.17).

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(b) Now let $n \geq 1$. Proposition 5.4 shows that (5.9) and (5.10) are valid. Because of $(s_j)_{j=0}^{2n} \in \tilde{\mathcal{D}}_{p \times q, 2n}$, we have $(s_j)_{j=0}^n \in \mathcal{D}_{p \times q, n}$. From (3.1), (5.1), and (2.7), we get $\mathbf{S}_n v_{q,n} = y_{0,n}$. Remark 4.7 yields $[R_{q,n}(\bar{\alpha})]^* \mathbf{S}_n^\dagger = \mathbf{S}_n^{\sharp, [\alpha]} = \mathbf{S}_n^\sharp [R_{p,n}(\bar{\alpha})]^*$. According to Proposition 3.5, we have $\mathbf{S}_n^\dagger \mathbf{S}_n = I_{n+1} \otimes (s_0^\dagger s_0)$. Using (5.1), (3.2), (3.1), and (3.5), we obtain $v_{q,n}^* \mathbf{S}_n^\sharp = z_{0,n}^\sharp$. Keeping in mind (5.1), we conclude $[I_{n+1} \otimes (s_0^\dagger s_0)] v_{q,n} = v_{q,n} s_0^\dagger s_0$. In view of (3.5) and Definition 3.1, we have $s_0^\dagger s_0 z_{0,n}^\sharp = z_{0,n}^\sharp$. Thus, we infer

$$\begin{aligned} \mathbf{S}_n^\dagger y_{0,n} v_{q,n}^* [R_{q,n}(\bar{\alpha})]^* \mathbf{S}_n^\dagger &= \mathbf{S}_n^\dagger \mathbf{S}_n v_{q,n} v_{q,n}^* \mathbf{S}_n^\sharp [R_{p,n}(\bar{\alpha})]^* \\ &= [I_{n+1} \otimes (s_0^\dagger s_0)] v_{q,n} z_{0,n}^\sharp [R_{p,n}(\bar{\alpha})]^* \\ &= v_{q,n} s_0^\dagger s_0 z_{0,n}^\sharp [R_{p,n}(\bar{\alpha})]^* = v_{q,n} z_{0,n}^\sharp [R_{p,n}(\bar{\alpha})]^*. \end{aligned} \quad (5.29)$$

Similarly, we see that

$$\mathbf{S}_n^\dagger R_{p,n}(\alpha) v_{p,n} z_{0,n} \mathbf{S}_n^\dagger = R_{q,n}(\alpha) y_{0,n}^\sharp v_{p,n}^* \quad (5.30)$$

holds true. According to Remark 4.2(b), we have $[R_{p,n}(\alpha)]^{-1} \mathbf{S}_n = \mathbf{S}_n [R_{q,n}(\alpha)]^{-1}$. Using (3.2), (3.1), (5.1), and (3.5), we get $\mathbf{S}_n^\sharp v_{p,n} = y_{0,n}^\sharp$. In view of (5.1), (3.1), and (2.7), we obtain $v_{p,n}^* \mathbf{S}_n = z_{0,n}$. Proposition 3.5 yields $\mathbf{S}_n \mathbf{S}_n^\dagger = \mathbf{S}_n \mathbf{S}_n^\dagger = I_{n+1} \otimes (s_0 s_0^\dagger)$. Keeping in mind (5.1), we conclude $[I_{n+1} \otimes (s_0 s_0^\dagger)] v_{p,n} = v_{p,n} s_0 s_0^\dagger$. Since the sequence $(s_j)_{j=0}^n$ belongs to $\mathcal{D}_{p \times q, n}$, we have, in view of (2.7), Definition 3.3, and Remark A.1(c), furthermore $s_0 s_0^\dagger z_{0,n} = z_{0,n}$. Thus, we infer

$$\begin{aligned} [R_{p,n}(\alpha)]^{-1} \mathbf{S}_n R_{q,n}(\alpha) y_{0,n}^\sharp v_{p,n}^* \mathbf{S}_n &= \mathbf{S}_n [R_{q,n}(\alpha)]^{-1} R_{q,n}(\alpha) \mathbf{S}_n^\sharp v_{p,n} z_{0,n} = \mathbf{S}_n \mathbf{S}_n^\sharp v_{p,n} z_{0,n} \\ &= [I_{n+1} \otimes (s_0 s_0^\dagger)] v_{p,n} z_{0,n} = v_{p,n} s_0 s_0^\dagger z_{0,n} = v_{p,n} z_{0,n}. \end{aligned}$$

Similarly, we get $\mathbf{S}_n v_{q,n} z_{0,n}^\sharp [R_{p,n}(\bar{\alpha})]^* \mathbf{S}_n [R_{q,n}(\bar{\alpha})]^{-*} = y_{0,n} v_{q,n}^*$. Hence,

$$\mathbf{S}_n \left(R_{q,n}(\alpha) y_{0,n}^\sharp v_{p,n}^* + v_{q,n} z_{0,n}^\sharp [R_{p,n}(\bar{\alpha})]^* \right) \mathbf{S}_n = y_{0,n} v_{q,n}^* [R_{q,n}(\bar{\alpha})]^* + R_{p,n}(\alpha) v_{p,n} z_{0,n}. \quad (5.31)$$

Applying (5.29) and (5.30) to equation (5.9), we see that (5.18) holds true, whereas (5.10) and (5.31) yield (5.19).

(c) Because of $s_0^{[\sharp, \alpha]} (s_{2n} - s_0 s_0^\dagger s_{2n} s_0^\dagger s_0) s_0^{[\sharp, \alpha]} = s_0^\dagger (s_{2n} - s_0 s_0^\dagger s_{2n} s_0^\dagger s_0) s_0^\dagger = 0_{q \times p}$, we have $\mathbf{S}_n^{[\sharp, \alpha]} \Xi_{n, 2n} \mathbf{S}_n^{[\sharp, \alpha]} = 0_{(n+1)q \times (n+1)p}$. Thus, (5.16) shows that

$$\text{rank } H_n^{[\sharp, \alpha]} \leq \text{rank} \left[H_n^{[+, \alpha]} - (y_{0,n}^{[+, \alpha]} v_{q,n}^* + v_{p,n} z_{0,n}^{[+, \alpha]}) - \Xi_{n, 2n} \right]$$

holds true. On the other hand, the inequality

$$\text{rank} \left[H_n^{[+, \alpha]} - (y_{0,n}^{[+, \alpha]} v_{q,n}^* + v_{p,n} z_{0,n}^{[+, \alpha]}) - \Xi_{n, 2n} \right] \leq \text{rank } H_n^{[\sharp, \alpha]}$$

immediately follows from (5.8).

(d) Part (d) is a direct consequence of (5.16) and $s_0^{[\sharp, \alpha]} = s_0^\dagger$. \square

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Theorem 5.6. *Let $\alpha \in \mathbb{C}$, let $n \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{2n+1} \in \tilde{\mathcal{D}}_{p \times q, 2n+1}$. Then:*

- (a) $K_n^{[\sharp, \alpha]} = -\mathbf{S}_n^{[\sharp, \alpha]} K_n^{[+, \alpha]} \mathbb{S}_n^{[\sharp, \alpha]}$ and $K_n^{[\sharp, \alpha]} = -R_{q,n}(\alpha) \mathbf{S}_n^\dagger H_{\alpha \triangleright n} \mathbb{S}_n^\dagger [R_{p,n}(\bar{\alpha})]^*$.
- (b) $K_n^{[+, \alpha]} = \Xi_{n, 2n+1} - \mathbf{S}_n^{[+, \alpha]} K_n^{[\sharp, \alpha]} \mathbb{S}_n^{[+, \alpha]}$.
- (c) $\text{rank } K_n^{[\sharp, \alpha]} = \text{rank}(K_n^{[+, \alpha]} - \Xi_{n, 2n+1})$.
- (d) If $p = q$, then $\det(K_n^{[\sharp, \alpha]}) = [(\det s_0)^\dagger]^{2n+2} \det(-K_n^{[+, \alpha]})$.

Proof. Remark 4.3(b) shows that $(s_j^{[+, \alpha]})_{j=0}^{2n+1}$ belongs to $\tilde{\mathcal{D}}_{p \times q, 2n+1}$. Thus, the first equation in (a) immediately follows from [25, Theorem 6.9(a)], whereas the second one is a consequence of this first equation, Remark 4.7, and Lemma 5.1(b). Since $(s_j^{[+, \alpha]})_{j=0}^{2n+1}$ belongs to $\tilde{\mathcal{D}}_{p \times q, 2n+1}$, from Remark A.1 we get

$$\begin{aligned} s_{2n+1}^{[+, \alpha]} - s_0^{[+, \alpha]} (s_0^{[+, \alpha]})^\dagger s_{2n+1}^{[+, \alpha]} (s_0^{[+, \alpha]})^\dagger s_0^{[+, \alpha]} &= -\alpha s_{2n} + s_{2n+1} - s_0 s_0^\dagger (-\alpha s_{2n} + s_{2n+1}) s_0^\dagger s_0 \\ &= s_{2n+1} - s_0 s_0^\dagger s_{2n+1} s_0^\dagger s_0. \end{aligned}$$

Consequently, using [25, Theorem 6.9], we get (b), (c), and (d). \square

Proposition 5.7. *Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n} \in \tilde{\mathcal{D}}_{p \times q, 2n}$. Then the block Hankel matrix G_{n-1}^\sharp admits the representation*

$$G_{n-1}^\sharp = -\nabla_{q,n}^* \mathbf{S}_n^\dagger H_n \mathbb{S}_n^\dagger \nabla_{p,n}. \quad (5.32)$$

Proof. From [25, Theorem 6.1(a)] we know that

$$H_n^\sharp = -\mathbf{S}_n^\dagger H_n \mathbb{S}_n^\dagger + y_{0,n}^\sharp v_{p,n}^* + v_{q,n} z_{0,n}^\sharp \quad (5.33)$$

holds true. Combining $\nabla_{q,n}^* H_n^\sharp \nabla_{p,n} = G_{n-1}^\sharp$, (5.33), and $v_{p,n}^* \nabla_{p,n} = 0_{p \times np}$, we obtain then (5.32). \square

Theorem 5.8. *Let $\alpha \in \mathbb{C}$, let $n \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{2n+2} \in \tilde{\mathcal{D}}_{p \times q, 2n+2}$. Then:*

- (a) $G_n^{[\sharp, \alpha]} = -\mathbf{S}_n^{[\sharp, \alpha]} \mathbb{L}_{n+1}^{[+, \alpha]} \mathbb{S}_n^{[\sharp, \alpha]}$ and

$$\begin{aligned} G_n^{[\sharp, \alpha]} &= -R_{q,n}(\alpha) \mathbf{S}_n^\dagger (K_{\alpha \triangleright n} - y_{\alpha \triangleright 0, n} s_0^\dagger z_{\alpha \triangleright 0, n}) \mathbb{S}_n^\dagger [R_{p,n}(\bar{\alpha})]^*, \\ G_n^{[\sharp, \alpha]} &= -\nabla_{q, n+1}^* \mathbf{S}_{n+1}^\dagger H_{n+1} \mathbb{S}_{n+1}^\dagger \nabla_{p, n+1} - \alpha R_{q,n}(\alpha) \mathbf{S}_n^\dagger H_{\alpha \triangleright n} \mathbb{S}_n^\dagger [R_{p,n}(\bar{\alpha})]^*. \end{aligned} \quad (5.34)$$
- (b) $G_n^{[+, \alpha]} = \Xi_{n, 2n+2} - \mathbf{S}_n^{[+, \alpha]} \mathbb{L}_{n+1}^{[\sharp, \alpha]} \mathbb{S}_n^{[+, \alpha]}$, where $\mathbb{L}_{n+1}^{[\sharp, \alpha]}$ is defined via (2.12) using the reciprocal sequence $(s_j^{[\sharp, \alpha]})_{j=0}^{2n+2}$ corresponding to the $[+, \alpha]$ -transform of $(s_j)_{j=0}^{2n+2}$.
- (c) $\text{rank } G_n^{[\sharp, \alpha]} = \text{rank}(\mathbb{L}_{n+1}^{[+, \alpha]} - \Xi_{n, 2n+2})$ and $\text{rank } G_n^{[\sharp, \alpha]} = \text{rank}(K_{\alpha \triangleright n} - y_{\alpha \triangleright 0, n} s_0^\dagger z_{\alpha \triangleright 0, n} - \Xi_{n, 2n+2})$.

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(d) If $p = q$, then $\det(G_n^{[\sharp, \alpha]}) = [(\det s_0)^\dagger]^{2n+2} \det(-\mathbb{L}_{n+1}^{[+, \alpha]})$ and $\det(G_n^{[\sharp, \alpha]}) = [(\det s_0)^\dagger]^{2n+2} \det[-(K_{\alpha \triangleright n} - y_{\alpha \triangleright 0, n} s_0^\dagger z_{\alpha \triangleright 0, n})]$.

Proof. The strategy of our proof is based on appropriate applications of [25, Theorem 6.13]. Denote by $(t_j)_{j=0}^{2n+2}$ the $[+, \alpha]$ -transform of $(s_j)_{j=0}^{2n+2}$. Remark 4.3(b) shows that

$$(t_j)_{j=0}^{2n+2} \in \tilde{\mathcal{D}}_{p \times q, 2n+2} \quad (5.35)$$

holds true. Using (2.12) and Lemma 5.1(d), we obtain

$$G_n^{(t)} - y_{1, n+1}^{(t)} t_0^\dagger z_{1, n+1}^{(t)} = \mathbb{L}_{n+1}^{(t)} = \mathbb{L}_{n+1}^{[+, \alpha]} = K_{\alpha \triangleright n} - y_{\alpha \triangleright 0, n} s_0^\dagger z_{\alpha \triangleright 0, n}. \quad (5.36)$$

Because of (5.35) and Definition 3.7, we have

$$(s_j)_{j=0}^{2n+1} \in \mathcal{D}_{p \times q, 2n+1} \quad \text{and} \quad (t_j)_{j=0}^{2n+1} \in \mathcal{D}_{p \times q, 2n+1}. \quad (5.37)$$

Denote by $(r_j)_{j=0}^{2n+2}$ the reciprocal sequence corresponding to $(t_j)_{j=0}^{2n+2}$. In view of (2.10), (4.4), (5.6), and (2.12), we get then

$$G_n^{(r)} = G_n^{[\sharp, \alpha]} \quad \text{and} \quad G_n^{(r)} - y_{1, n+1}^{(r)} r_0^\dagger z_{1, n+1}^{(r)} = \mathbb{L}_{n+1}^{(r)} = \mathbb{L}_{n+1}^{[\sharp, \alpha]}. \quad (5.38)$$

Because of (5.37) and Remark 4.7, we have furthermore

$$\mathbf{S}_n^{(r)} = \mathbf{S}_n^{[\sharp, \alpha]} = [R_{q, n}(\alpha)] \mathbf{S}_n^\dagger \quad \text{and} \quad \mathbb{S}_n^{(r)} = \mathbb{S}_n^{[\sharp, \alpha]} = \mathbb{S}_n^\dagger [R_{p, n}(\overline{\alpha})]^*. \quad (5.39)$$

According to (5.35), we can apply [25, Theorem 6.13(a)] to the sequence $(t_j)_{j=0}^{2n+2}$ and obtain, by means of (5.38), (5.39), and (5.36), the first two equations stated in (a).

In view of (5.35), the application of Proposition 5.7 to the sequence $(t_j)_{j=0}^{2n+2}$ yields

$$G_n^{(r)} = -\nabla_{q, n+1}^* (\mathbf{S}_{n+1}^{(t)})^\dagger H_{n+1}^{(t)} (\mathbb{S}_{n+1}^{(t)})^\dagger \nabla_{p, n+1}. \quad (5.40)$$

Because of (5.37), Proposition 3.5 provides us

$$\mathbf{S}_m^\dagger = \mathbf{S}_m^\sharp, \quad \mathbb{S}_m^\dagger = \mathbb{S}_m^\sharp, \quad (\mathbf{S}_m^{(t)})^\dagger = \mathbf{S}_m^{(r)}, \quad \text{and} \quad (\mathbb{S}_m^{(t)})^\dagger = \mathbb{S}_m^{(r)} \quad (5.41)$$

for all $m \in \mathbb{Z}_{0, 2n+1}$. Keeping in mind (5.1), (3.2), (3.1), and (3.5), this implies

$$\nabla_{q, n+1}^* \mathbf{S}_{n+1}^\dagger = [y_{1, n+1}^\sharp, \mathbf{S}_n^\dagger] \quad \text{and} \quad \mathbb{S}_{n+1}^\dagger \nabla_{p, n+1} = \begin{bmatrix} z_{1, n+1}^\sharp \\ \mathbb{S}_n^\dagger \end{bmatrix}. \quad (5.42)$$

According to (5.41) and (5.37), Remark 4.7 shows that

$$(\mathbf{S}_{n+1}^{(t)})^\dagger = \mathbf{S}_{n+1}^\dagger [R_{p, n+1}(\alpha)], \quad (\mathbb{S}_{n+1}^{(t)})^\dagger = [R_{q, n+1}(\overline{\alpha})]^* \mathbb{S}_{n+1}^\dagger \quad (5.43)$$

and

$$\mathbf{S}_n^\dagger [R_{p, n}(\alpha)] = [R_{q, n}(\alpha)] \mathbf{S}_n^\dagger, \quad [R_{q, n}(\overline{\alpha})]^* \mathbb{S}_n^\dagger = \mathbb{S}_n^\dagger [R_{p, n}(\overline{\alpha})]^* \quad (5.44)$$

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hold true. By virtue of Lemma 5.1(a), the matrix $H_{n+1}^{(t)}$ can be represented via

$$H_{n+1}^{(t)} = [R_{p,n+1}(\alpha)]^{-1} H_{n+1} [R_{q,n+1}(\bar{\alpha})]^{-*} + \alpha \nabla_{p,n+1} H_{\alpha \triangleright n} \nabla_{q,n+1}^*$$

Using additionally (5.38), (5.40), and (5.43), we conclude

$$\begin{aligned} G_n^{[\sharp, \alpha]} &= G_n^{(r)} = -\nabla_{q,n+1}^* \mathbf{S}_{n+1}^\dagger [R_{p,n+1}(\alpha)] H_{n+1}^{(t)} [R_{q,n+1}(\bar{\alpha})]^* \mathbb{S}_{n+1}^\dagger \nabla_{p,n+1} \\ &= -\nabla_{q,n+1}^* \mathbf{S}_{n+1}^\dagger H_{n+1} \mathbb{S}_{n+1}^\dagger \nabla_{p,n+1} \\ &\quad - \alpha \nabla_{q,n+1}^* \mathbf{S}_{n+1}^\dagger [R_{p,n+1}(\alpha)] \nabla_{p,n+1} H_{\alpha \triangleright n} \nabla_{q,n+1}^* [R_{q,n+1}(\bar{\alpha})]^* \mathbb{S}_{n+1}^\dagger \nabla_{p,n+1}. \end{aligned} \quad (5.45)$$

From (4.3) and (5.1) we get $[R_{p,n+1}(\alpha)] \nabla_{p,n+1} = \begin{bmatrix} 0_{p \times (n+1)p} \\ R_{p,n}(\alpha) \end{bmatrix}$ and $\nabla_{q,n+1}^* [R_{q,n+1}(\bar{\alpha})]^* = [0_{(n+1)q \times q}, [R_{q,n}(\bar{\alpha})]^*]$. Using additionally (5.42), and (5.44), we obtain

$$\nabla_{q,n+1}^* \mathbf{S}_{n+1}^\dagger [R_{p,n+1}(\alpha)] \nabla_{p,n+1} = \mathbf{S}_n^\dagger [R_{p,n}(\alpha)] = [R_{q,n}(\alpha)] \mathbf{S}_n^\dagger \quad (5.46)$$

and

$$\nabla_{q,n+1}^* [R_{q,n+1}(\bar{\alpha})]^* \mathbb{S}_{n+1}^\dagger \nabla_{p,n+1} = [R_{q,n}(\bar{\alpha})]^* \mathbb{S}_n^\dagger = \mathbb{S}_n^\dagger [R_{p,n}(\bar{\alpha})]^*. \quad (5.47)$$

Thus, equation (5.34) follows from (5.45), (5.46), and (5.47).

In view of (4.1), Lemma 5.1(c), and the notations given in Remark 4.2(b), we have

$$t_0 = s_0, \quad G_n^{(t)} = G_n^{[+, \alpha]}, \quad \mathbf{S}_n^{(t)} = \mathbf{S}_n^{[+, \alpha]}, \quad \text{and} \quad \mathbb{S}_n^{(t)} = \mathbb{S}_n^{[+, \alpha]}. \quad (5.48)$$

According to (5.37), parts (c) and (b) of Remark A.1 yield $s_0 s_0^\dagger s_{2n+1} s_0^\dagger s_0 = s_{2n+1}$. By virtue of Definition 4.1, we get hence

$$\begin{aligned} t_{2n+2} - t_0 t_0^\dagger t_{2n+2} t_0^\dagger t_0 &= (-\alpha s_{2n+1} + s_{2n+2}) - s_0 s_0^\dagger (-\alpha s_{2n+1} + s_{2n+2}) s_0^\dagger s_0 \\ &= s_{2n+2} - s_0 s_0^\dagger s_{2n+2} s_0^\dagger s_0, \end{aligned}$$

which, in view of (5.4), implies

$$\Xi_{n, 2n+2}^{(t)} = \Xi_{n, 2n+2}. \quad (5.49)$$

Because of (5.35), we can apply parts (b)–(d) of [25, Theorem 6.13] to the sequence $(t_j)_{j=0}^{2n+2}$ and obtain, by means of (5.48), (5.36), (5.38), and (5.49), consequently (b)–(d). \square

6. The shortened negative reciprocal sequence corresponding to the $[+, \alpha]$ -transform of a sequence from $\mathcal{D}_{p \times q, \kappa}$

This section should be compared with [25, Section 7]. The main goal of this section is to prepare the elementary step of the later Schur-type algorithm. As in [25, Section 7], the main tool will be to use appropriately chosen reciprocal sequences. However, in [25] a

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two-step algorithm was applied. In this section, we develop a one-step algorithm where, in addition, the influence of the given real number α has to be regarded.

Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then, using Definition 4.1 and (4.4), let the sequence $(s_j^{(1, \alpha)})_{j=0}^{\kappa-1}$ be defined by

$$s_j^{(1, \alpha)} := -s_{j+1}^{[\sharp, \alpha]} \quad (6.1)$$

for all $j \in \mathbb{Z}_{0, \kappa-1}$. Now we state some observations on the arithmetics of the transform just introduced.

Remark 6.1. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. In view of (6.1), (4.4), Definition 4.1, and Remark 3.2, one can easily see that, for all $m \in \mathbb{Z}_{0, \kappa-1}$, the sequence $(s_j^{(1, \alpha)})_{j=0}^m$ depends only on the matrices s_0, s_1, \dots, s_{m+1} .

Remark 6.2. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. In view of (6.1), Definition 3.3, [26, Proposition 5.10(a)], (4.4), Definition 3.1, (4.1), and Remark A.1(a), then $\bigcup_{j=0}^{\kappa-1} \mathcal{R}(s_j^{(1, \alpha)}) \subseteq \mathcal{R}(s_0^*)$ and $\mathcal{N}(s_0^*) \subseteq \bigcap_{j=0}^{\kappa-1} \mathcal{N}(s_j^{(1, \alpha)})$.

Remark 6.3. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. In view of (6.1) and Lemma 4.11, then:

- (a) If $\gamma \in \mathbb{C}$, then $((\gamma s_j)^{(1, \alpha)})_{j=0}^{\kappa-1} = (\gamma^\dagger s_j^{(1, \alpha)})_{j=0}^{\kappa-1}$ and $((\gamma^j s_j)^{(1, \alpha)})_{j=0}^{\kappa-1} = (\gamma^{j+1} s_j^{(1, \alpha)})_{j=0}^{\kappa-1}$.
- (b) If $m \in \mathbb{N}$ and $L \in \mathbb{C}^{m \times p}$ with $\mathcal{R}(L^*) = \mathcal{R}(s_0)$, then $((L s_j)^{(1, \alpha)})_{j=0}^{\kappa-1} = (s_j^{(1, \alpha)} L^\dagger)_{j=0}^{\kappa-1}$.
- (c) If $n \in \mathbb{N}$ and $R \in \mathbb{C}^{q \times n}$ with $\mathcal{R}(s_0^*) = \mathcal{R}(R)$, then $((s_j R)^{(1, \alpha)})_{j=0}^{\kappa-1} = (R^\dagger s_j^{(1, \alpha)})_{j=0}^{\kappa-1}$.
- (d) If $m, n \in \mathbb{N}$, $L \in \mathbb{C}^{m \times p}$ with $\mathcal{R}(L^*) = \mathcal{R}(s_0)$ and $R \in \mathbb{C}^{q \times n}$ with $\mathcal{R}(s_0^*) = \mathcal{R}(R)$, then $((L s_j R)^{(1, \alpha)})_{j=0}^{\kappa-1} = (R^\dagger s_j^{(1, \alpha)} L^\dagger)_{j=0}^{\kappa-1}$.
- (e) If $m, n \in \mathbb{N}$, $U \in \mathbb{C}^{m \times p}$ with $U^* U = I_p$ and $V \in \mathbb{C}^{q \times n}$ with $V V^* = I_q$, then $((U s_j V)^{(1, \alpha)})_{j=0}^{\kappa-1} = (V^* s_j^{(1, \alpha)} U^*)_{j=0}^{\kappa-1}$.
- (f) If the sequence $(t_j)_{j=0}^\kappa$ is given by $t_j := s_j^*$ for all $j \in \mathbb{Z}_{0, \kappa}$, then $(s_j^{(1, \alpha)})^* = t_j^{(1, \bar{\alpha})}$ for all $j \in \mathbb{Z}_{0, \kappa-1}$.

Remark 6.4. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $n \in \mathbb{N}$. For all $m \in \mathbb{Z}_{1, n}$, let $p_m, q_m \in \mathbb{N}$ and let $(s_j^{(m)})_{j=0}^\kappa$ be a sequence of complex $p_m \times q_m$ matrices. In view of (6.1) and Remark 4.12, then $(\text{diag}[(s_j^{(m)})^{(1, \alpha)}]_{m=1}^n)_{j=0}^{\kappa-1} = ((\text{diag}[s_j^{(m)}]_{m=1}^n)^{(1, \alpha)})_{j=0}^{\kappa-1}$.

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Remark 6.5. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. In view of (6.1), (4.4), Definition 3.1, (4.1), and Lemma 4.6, then it is readily checked that

$$s_j^{(1, \alpha)} = s_0^\dagger \sum_{l=0}^j s_{j+1-l}^{[+, \alpha]} s_l^{[\sharp, \alpha]} \quad \text{and} \quad s_j^{(1, \alpha)} = - \sum_{l=0}^{j+1} \alpha^{j+1-l} s_l^\sharp \quad \text{for all } j \in \mathbb{Z}_{0, \kappa-1}.$$

Remark 6.6. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then, in view of Remark 6.5, (4.4), Definition 3.1, (4.1), and (6.1), one can easily see that

$$s_0^{(1, \alpha)} = s_0^\dagger s_1^{[+, \alpha]} s_0^\dagger \quad \text{and} \quad s_j^{(1, \alpha)} = s_0^\dagger s_{j+1}^{[+, \alpha]} s_0^\dagger - s_0^\dagger \sum_{l=0}^{j-1} s_{j-l}^{[+, \alpha]} s_l^{(1, \alpha)} \quad \text{for all } j \in \mathbb{Z}_{1, \kappa-1}.$$

Lemma 6.7. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. For all $j \in \mathbb{Z}_{1, \kappa}$, then*

$$s_0^\dagger s_j^{[+, \alpha]} s_0^\dagger = s_0^\dagger \sum_{l=0}^{j-1} s_{j-1-l}^{[+, \alpha]} s_l^{(1, \alpha)}.$$

Proof. In view of (4.1) and Remark 6.6, we have $s_0^\dagger s_0^{[+, \alpha]} s_l^{(1, \alpha)} = s_l^{(1, \alpha)}$ for all $l \in \mathbb{Z}_{0, j-1}$. Hence, Remark 6.6 yields $s_0^\dagger s_0^{[+, \alpha]} s_0^{(1, \alpha)} = s_0^\dagger s_1^{[+, \alpha]} s_0^\dagger$ and, in the case $\kappa \geq 2$, for all $j \in \mathbb{Z}_{2, \kappa}$, furthermore

$$\begin{aligned} s_0^\dagger \sum_{l=0}^{j-1} s_{j-1-l}^{[+, \alpha]} s_l^{(1, \alpha)} &= s_0^\dagger \sum_{l=0}^{j-2} s_{j-1-l}^{[+, \alpha]} s_l^{(1, \alpha)} + s_0^\dagger s_0^{[+, \alpha]} s_{j-1}^{(1, \alpha)} \\ &= s_0^\dagger s_j^{[+, \alpha]} s_0^\dagger - s_{j-1}^{(1, \alpha)} + s_{j-1}^{(1, \alpha)} = s_0^\dagger s_j^{[+, \alpha]} s_0^\dagger. \end{aligned} \quad \square$$

Lemma 6.8. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ and $(t_j)_{j=0}^\kappa$ be sequences of complex $p \times q$ matrices. Then the following statements are equivalent:*

- (i) $s_j^{(1, \alpha)} = t_j^{(1, \alpha)}$ for all $j \in \mathbb{Z}_{0, \kappa-1}$ and $s_0 = t_0$.
- (ii) $s_0 s_0^\dagger s_j s_0^\dagger s_0 = t_0 t_0^\dagger t_j t_0^\dagger t_0$ for all $j \in \mathbb{Z}_{0, \kappa}$.

Proof. According to (4.1), Definition 3.1, (4.4), and Remark A.1(a), statement (i) is equivalent to:

- (iii) $s_j^{(1, \alpha)} = t_j^{(1, \alpha)}$ for all $j \in \mathbb{Z}_{0, \kappa-1}$ and $s_0^{[\sharp, \alpha]} = t_0^{[\sharp, \alpha]}$.

In view of (6.1) and Lemma 4.10, statement (iii) is equivalent to (ii). \square

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Remark 6.9. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, let $(s_j)_{j=0}^\kappa$ be a sequence from $\mathbb{C}^{p \times q}$, and let $n \in \mathbb{Z}_{0, \kappa-1}$. In view of (3.1), (6.1), Remark 4.7, (5.1) and (4.2), then

$$\begin{aligned} \mathbf{S}_n^{(s^{(1, \alpha)})} &= -\nabla_{q, n+1}^* \left([R_{q, n+1}(\alpha)] \mathbf{S}_{n+1}^\# - [I_{n+2} \otimes (s_0^\dagger)] \right) \Delta_{p, n+1} \\ &= \left[T_{1, n}^* \otimes (s_0^\dagger) \right] - \nabla_{q, n+1}^* [R_{q, n+1}(\alpha)] \mathbf{S}_{n+1}^\# \Delta_{p, n+1} \end{aligned}$$

and

$$\begin{aligned} \mathbb{S}_n^{(s^{(1, \alpha)})} &= -\Delta_{q, n+1}^* \left(\mathbb{S}_{n+1}^\# [R_{p, n+1}(\bar{\alpha})]^* - [I_{n+2} \otimes (s_0^\dagger)] \right) \nabla_{p, n+1} \\ &= \left[T_{1, n} \otimes (s_0^\dagger) \right] - \Delta_{q, n+1}^* \mathbb{S}_{n+1}^\# [R_{p, n+1}(\bar{\alpha})]^* \nabla_{p, n+1}. \end{aligned}$$

Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence from $\mathbb{C}^{p \times q}$. Then, let

$$H_n^{(1, \alpha)} := [s_{j+k}^{(1, \alpha)}]_{j, k=0}^n \quad (6.2)$$

for all $n \in \mathbb{N}_0$ with $2n \leq \kappa - 1$ and let

$$K_n^{(1, \alpha)} := [s_{j+k+1}^{(1, \alpha)}]_{j, k=0}^n \quad (6.3)$$

for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa - 1$. Now we derive some formulas for several block Hankel matrices associated with the sequence introduced in (6.1).

Lemma 6.10. *Let $\alpha \in \mathbb{C}$, let $n \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{2n+1} \in \tilde{\mathcal{D}}_{p \times q, 2n+1}$. Then:*

- (a) $H_n^{(1, \alpha)} = -K_n^{[\sharp, \alpha]} = [R_{q, n}(\alpha)] \mathbf{S}_n^\dagger H_{\alpha \triangleright n} \mathbb{S}_n^\dagger [R_{p, n}(\bar{\alpha})]^*$.
- (b) $\text{rank}(H_n^{(1, \alpha)}) = \text{rank}(H_{\alpha \triangleright n} - \Xi_{n, 2n+1})$.
- (c) If $p = q$, then $\det(H_n^{(1, \alpha)}) = [(\det s_0)^\dagger]^{2n+2} \det H_{\alpha \triangleright n}$.

Proof. (a) Use (6.2), (6.1), (5.5) and Theorem 5.6(a).

(b) Use the first equation in (a), Theorem 5.6(c) and Lemma 5.1(b).

(c) Use the first equation in (a), Theorem 5.6(d) and Lemma 5.1(b). \square

Lemma 6.11. *Let $\alpha \in \mathbb{C}$, let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n} \in \tilde{\mathcal{D}}_{p \times q, 2n}$. Then:*

- (a) $-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)} = \nabla_{q, n}^* \mathbf{S}_n^\dagger H_n \mathbb{S}_n^\dagger \nabla_{p, n}$ and $-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)} = \mathbf{S}_{n-1}^\dagger \mathbb{L}_n \mathbb{S}_{n-1}^\dagger$.
- (b) $\text{rank}(-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)}) = \text{rank}(H_n - \Xi_{n, 2n}) - \text{rank } s_0$ and $\text{rank}(-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)}) = \text{rank}(\mathbb{L}_n - \Xi_{n-1, 2n})$.
- (c) $\det(-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)}) = [(\det s_0)^\dagger]^{2n+1} \det H_n$ and $\det(-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)}) = [(\det s_0)^\dagger]^{2n} \det \mathbb{L}_n$.

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Proof. (a) In view of (6.3), (6.1) and (5.6), we have $K_{n-1}^{(1, \alpha)} = -G_{n-1}^{[\sharp, \alpha]}$. Taking additionally into account Lemma 6.10(a), we obtain $-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)} = \alpha K_{n-1}^{[\sharp, \alpha]} - G_{n-1}^{[\sharp, \alpha]}$. Hence, the first equation in (a) is an immediate consequence of Theorem 5.6(a) and Theorem 5.8(a). From [25, Theorem 6.1(a)] we know that $H_n^\sharp + \mathbf{S}_n^\dagger H_n \mathbf{S}_n^\dagger = y_{0,n}^\sharp v_{p,n}^* + v_{q,n} z_{0,n}^\sharp$. In view of (5.1), we have $\nabla_{q,n}^*(y_{0,n}^\sharp v_{p,n}^* + v_{q,n} z_{0,n}^\sharp) \nabla_{p,n} = 0$ and, consequently,

$$G_{n-1}^\sharp = \nabla_{q,n}^* H_n^\sharp \nabla_{p,n} = -\nabla_{q,n}^* \mathbf{S}_n^\dagger H_n \mathbf{S}_n^\dagger \nabla_{p,n}. \quad (6.4)$$

Furthermore, [25, Theorem 6.13(a)] and (2.12) yield $G_{n-1}^\sharp = -\mathbf{S}_{n-1}^\dagger \mathbb{L}_n \mathbf{S}_{n-1}^\dagger$. Thus, (6.4) and the first equation in (a) yield the second one.

(b) Obviously, $s_0^\dagger (s_{2n} - s_0 s_0^\dagger s_{2n} s_0^\dagger s_0) s_0^\dagger = 0$. Since Proposition 3.5 yields

$$\mathbf{S}_k^\dagger = \mathbf{S}_k^\sharp \quad \text{and} \quad \mathbb{S}_k^\dagger = \mathbb{S}_k^\sharp \quad \text{for all } k \in \mathbb{Z}_{0,n}, \quad (6.5)$$

we then have $\mathbf{S}_{n-1}^\dagger \Xi_{n-1,2n} \mathbf{S}_{n-1}^\dagger = 0$. In view of part (a), this implies

$$-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)} = \mathbf{S}_{n-1}^\dagger (\mathbb{L}_n - \Xi_{n-1,2n}) \mathbf{S}_{n-1}^\dagger. \quad (6.6)$$

In particular,

$$\text{rank}(-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)}) \leq \text{rank}(\mathbb{L}_n - \Xi_{n-1,2n}). \quad (6.7)$$

Multiplying equation (6.6) from the left by \mathbf{S}_{n-1} and from the right by \mathbb{S}_{n-1} and using Proposition 3.5, we conclude

$$\mathbf{S}_{n-1}(-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)}) \mathbb{S}_{n-1} = \left[I_n \otimes (s_0 s_0^\dagger) \right] (\mathbb{L}_n - \Xi_{n-1,2n}) \left[I_n \otimes (s_0^\dagger s_0) \right]. \quad (6.8)$$

Since $(s_j)_{j=0}^{2n}$ belongs to $\tilde{\mathcal{D}}_{p \times q, 2n}$, from parts (c) and (b) of Remark A.1 we get $s_0 s_0^\dagger s_j = s_j$ and $s_j s_0^\dagger s_0 = s_j$ hold for all $j \in \mathbb{Z}_{0,2n-1}$. Because the $q \times q$ block in the right lower corner of $\mathbb{L}_n - \Xi_{n-1,2n}$ is exactly $s_0 s_0^\dagger s_{2n} s_0^\dagger s_0 - s_n s_0^\dagger s_n$, then from (2.12) it follows that

$$\left[I_n \otimes (s_0 s_0^\dagger) \right] (\mathbb{L}_n - \Xi_{n-1,2n}) \left[I_n \otimes (s_0^\dagger s_0) \right] = \mathbb{L}_n - \Xi_{n-1,2n}.$$

Combining this with (6.8), we get

$$\text{rank}(-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)}) \geq \text{rank}(\mathbb{L}_n - \Xi_{n-1,2n}). \quad (6.9)$$

From (6.7) and (6.9) we infer the second equation in (b). Taking into account that

$$H_n - \Xi_{n,2n} = \begin{bmatrix} s_0 & z_{1,n} \\ y_{1,n} & G_{n-1} - \Xi_{n-1,2n} \end{bmatrix}$$

holds true and that $(s_j)_{j=0}^{2n} \in \tilde{\mathcal{D}}_{p \times q, 2n}$ implies $\mathcal{N}(s_0) \subseteq \mathcal{N}(y_{1,n})$ and $\mathcal{R}(z_{1,n}) \subseteq \mathcal{R}(s_0)$, we get from [11, Lemma 1.1.7(a)] that

$$\text{rank}(H_n - \Xi_{n,2n}) = \text{rank } s_0 + \text{rank}(\mathbb{L}_n - \Xi_{n-1,2n}). \quad (6.10)$$

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Similarly, we see that

$$\det H_n = \det(s_0) \det(\mathbb{L}_n) \quad (6.11)$$

is valid. The second equation in (b) and (6.10) imply the first equation in (b).

(c) Use the second equation in (a) and (6.5) to get the second equation. The first equation follows in the case $\det s_0 \neq 0$ from the second one and (6.11) and in the case $\det s_0 = 0$ from the second one and $n \geq 1$. \square

Lemma 6.12. *Let $\alpha \in \mathbb{C}$, let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n} \in \tilde{\mathcal{D}}_{p \times q, 2n}$. Then*

$$\mathbf{S}_n^\dagger H_n \mathbf{S}_n^\dagger = \text{diag}[s_0^\sharp, -\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)}]. \quad (6.12)$$

Proof. From [25, Theorem 6.1(a)] we get $H_n^\sharp + \mathbf{S}_n^\dagger H_n \mathbf{S}_n^\dagger = y_{0,n}^\sharp v_{p,n}^* + v_{q,n} z_{0,n}^\sharp$. In view of (3.3), (3.4), (3.5), and (5.1), this implies $\mathbf{S}_n^\dagger H_n \mathbf{S}_n^\dagger = \text{diag}[s_0^\sharp, -G_{n-1}^\sharp]$. Taking additionally into account Lemma 6.11(a) and (5.1), we obtain in particular

$$-G_{n-1}^\sharp = \nabla_{q,n}^* \mathbf{S}_n^\dagger H_n \mathbf{S}_n^\dagger \nabla_{p,n} = -\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)}. \quad \square$$

The following proposition plays a key role in our further considerations. In its proof we will make essential use of Lemmas 6.10 and 6.11.

Proposition 6.13. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. Then:*

- (a) *If $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^\geq$, then $(s_j^{(1, \alpha)})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^\geq$.*
- (b) *If $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$, then $(s_j^{(1, \alpha)})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^{\geq, e}$.*
- (c) *If $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^>$, then $(s_j^{(1, \alpha)})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^>$.*
- (d) *If $m \in \mathbb{Z}_{0, \kappa}$ and $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, \text{cd}, m}$, then $(s_j^{(1, \alpha)})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^{\geq, \text{cd}, \max\{0, m-1\}}$.*
- (e) *If $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, \text{cd}}$, then $(s_j^{(1, \alpha)})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^{\geq, \text{cd}}$.*

Proof. (a) Let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^\geq$. From Proposition 3.8(c) we then know that $(s_j)_{j=0}^\kappa \in \tilde{\mathcal{D}}_{q \times q, \kappa}$. Consequently, for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, Lemma 6.10(a) yields

$$H_n^{(1, \alpha)} = [R_{q,n}(\alpha)] \mathbf{S}_n^\dagger H_{\alpha \triangleright n} \mathbf{S}_n^\dagger [R_{q,n}(\alpha)]^*. \quad (6.13)$$

Lemma 6.11(a) shows that, for all $n \in \mathbb{N}$ with $2n \leq \kappa$ the equation

$$-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)} = \nabla_{q,n}^* \mathbf{S}_n^\dagger H_n \mathbf{S}_n^\dagger \nabla_{q,n} \quad (6.14)$$

holds true. Since $(s_j)_{j=0}^\kappa$ belongs to $\mathcal{K}_{q, \kappa, \alpha}^\geq$, we see from Lemma 2.3(a), that $s_j^* = s_j$ holds for all $j \in \mathbb{Z}_{0, \kappa}$, which implies $\mathbf{S}_n^\dagger = (\mathbf{S}_n^\dagger)^*$ for all $n \in \mathbb{Z}_{0, \kappa}$. Because of $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^\geq$, we also have $H_n \in \mathbb{C}_{\geq}^{(n+1)q \times (n+1)q}$ for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$ and $H_{\alpha \triangleright n} \in \mathbb{C}_{\geq}^{(n+1)q \times (n+1)q}$

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for all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$. Thus, we conclude that, for all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, the right-hand side of (6.13) is non-negative Hermitian and that, for all $n \in \mathbb{N}$ with $2n \leq \kappa$, the right-hand side of (6.14) is non-negative Hermitian. Consequently, $H_n^{(1, \alpha)} \in \mathbb{C}_{\geq}^{(n+1)q \times (n+1)q}$ for all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$ and $-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)} \in \mathbb{C}_{\geq}^{nq \times nq}$ for all $n \in \mathbb{N}$ with $2n \leq \kappa$. Hence $(s_j^{(1, \alpha)})_{j=0}^{\kappa-1}$ belongs to $\mathcal{K}_{q, \kappa-1, \alpha}^{\geq}$.

(b) In the case $\kappa = +\infty$, part (b) is already proved in part (a). Now let $m \in \mathbb{N}_0$ and let $(s_j)_{j=0}^m \in \mathcal{K}_{q, m, \alpha}^{\geq, e}$. Then there is an $s_{m+1} \in \mathbb{C}^{q \times q}$ such that $(s_j)_{j=0}^{m+1} \in \mathcal{K}_{q, m+1, \alpha}^{\geq}$. According to part (a), then we see that $(s_j^{(1, \alpha)})_{j=0}^m$ belongs to $\mathcal{K}_{q, m, \alpha}^{\geq}$, which implies $(s_j^{(1, \alpha)})_{j=0}^{m-1} \in \mathcal{K}_{q, m-1, \alpha}^{\geq, e}$. Thus, in view of Remark 6.1, part (b) is proved.

(c) Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q, \kappa, \alpha}^{\geq}$. Parts (d) and (a) of Proposition 3.8 then yields $(s_j)_{j=0}^{\kappa} \in \mathcal{D}_{q \times q, \kappa}$, which, in view of the Definitions 3.3 and 3.7, implies $(s_j)_{j=0}^m \in \tilde{\mathcal{D}}_{q \times q, m}$ for all $m \in \mathbb{Z}_{0, \kappa}$. Parts (d) and (c) of Proposition 3.8 and (a) show that $(s_j^{(1, \alpha)})_{j=0}^{\kappa-1}$ belongs to $\mathcal{K}_{q, \kappa-1, \alpha}^{\geq}$. In other words, we have:

(I) For all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, the matrix $H_n^{(1, \alpha)}$ is non-negative Hermitian.

and

(II) For all $n \in \mathbb{N}$ with $2n \leq \kappa$, the matrix $-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)}$ is non-negative Hermitian.

Since $(s_j)_{j=0}^{\kappa}$ belongs to $\mathcal{K}_{q, \kappa, \alpha}^{\geq}$, the following two statements hold true:

(III) For all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, the matrix H_n is positive Hermitian.

(IV) For all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, the matrix $H_{\alpha \triangleright n}$ is positive Hermitian.

Using Lemma 6.10(c) and (IV), we see that:

(V) For all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, the matrix $H_n^{(1, \alpha)}$ is non-singular.

From (I) and (V) we see that:

(VI) For all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa$, the matrix $H_n^{(1, \alpha)}$ is positive Hermitian.

Using Lemma 6.11(c) and (III), we get that:

(VII) For all $n \in \mathbb{N}$ with $2n \leq \kappa$, the matrix $-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)}$ is non-singular.

From (II) and (VII) we obtain that:

(VIII) For all $n \in \mathbb{N}$ with $2n \leq \kappa$, the matrix $-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)}$ is positive Hermitian.

Because of (VI) and (VIII), the sequence $(s_j^{(1, \alpha)})_{j=0}^{\kappa-1}$ belongs to $\mathcal{K}_{q, \kappa-1, \alpha}^{\geq}$.

(d) Let $m \in \mathbb{Z}_{0, \kappa}$ and let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, \text{cd}, m}$. Because of (2.5), then $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q, \kappa, \alpha}^{\geq}$, which, in view of (a), implies $(s_j^{(1, \alpha)})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^{\geq}$. Furthermore, (2.5) yields $(s_j)_{j=0}^m \in \mathcal{K}_{q, m, \alpha}^{\geq, \text{cd}}$. According to parts (d) and (a) of Proposition 3.8, we have then $(s_j)_{j=0}^m \in \mathcal{D}_{q \times q, m}$,

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which, in view of the Definitions 3.3 and 3.7, implies $(s_j)_{j=0}^l \in \tilde{\mathcal{D}}_{q \times q, l}$ for all $l \in \mathbb{Z}_{0, m}$ and $s_0 s_0^\dagger s_j = s_j$ and $s_j s_0^\dagger s_0 = s_j$ for all $j \in \mathbb{Z}_{0, m}$. In view of (5.4), this implies

$$\Xi_{n, l} = 0 \quad \text{for all } n \in \mathbb{N}_0 \text{ and all } l \in \mathbb{Z}_{0, m}. \quad (6.15)$$

If $m = 0$, then $s_0 = 0_{q \times q}$. Thus, in view of Remark 6.2, the assertion holds true in the case $m = 0$.

We now consider the case that $m = 2n + 1$ with some $n \in \mathbb{N}_0$. Because of (2.4), then $(s_{\alpha \triangleright j})_{j=0}^{2n} \in \mathcal{H}_{q, 2n}^{\geq, \text{cd}}$, which, in view of (2.2), implies $L_{\alpha \triangleright n} = 0_{q \times q}$ and that the matrix $H_{\alpha \triangleright n}$ is non-negative Hermitian. If $n \geq 1$, then $H_{\alpha \triangleright n}$ admits the block representation

$$H_{\alpha \triangleright n} = \begin{bmatrix} H_{\alpha \triangleright n-1} & y_{\alpha \triangleright n, 2n-1} \\ z_{\alpha \triangleright n, 2n-1} & s_{\alpha \triangleright 2n} \end{bmatrix},$$

which, in view of [11, Lemmas 1.1.9 and 1.1.7], implies $\text{rank } H_{\alpha \triangleright n} = \text{rank } H_{\alpha \triangleright n-1} + \text{rank } L_{\alpha \triangleright n}$. Since $H_{\alpha \triangleright 0} = s_{\alpha \triangleright 0} = L_{\alpha \triangleright 0}$ holds true, we have thus $\text{rank } H_{\alpha \triangleright n} = 0$ in the case $n = 0$ and, in the case $n \geq 1$, furthermore $\text{rank } H_{\alpha \triangleright n} = \text{rank } H_{\alpha \triangleright n-1}$. Because of Lemma 6.10(b), (6.15), and Remark 6.1, this implies $\text{rank } H_n^{(1, \alpha)} = 0$ in the case $n = 0$ and, in the case $n \geq 1$, furthermore $\text{rank } H_n^{(1, \alpha)} = \text{rank } H_{n-1}^{(1, \alpha)}$. Because of Remark 2.1 and (1.4), we have $(s_j^{(1, \alpha)})_{j=0}^{2n} \in \mathcal{K}_{q, 2n, \alpha}^{\geq} \subseteq \mathcal{H}_{q, 2n}^{\geq}$. Thus, similar to the considerations above, from [11, Lemmas 1.1.9 and 1.1.7], we conclude that $\text{rank } L_n^{(s^{(1, \alpha)})} = 0$ and hence $L_n^{(s^{(1, \alpha)})} = 0_{q \times q}$. According to (2.2), we obtain then $(s_j^{(1, \alpha)})_{j=0}^{2n} \in \mathcal{H}_{q, 2n}^{\geq, \text{cd}}$ and, in view of (2.3), consequently $(s_j^{(1, \alpha)})_{j=0}^{2n} \in \mathcal{K}_{q, 2n, \alpha}^{\geq, \text{cd}}$.

Now we consider the case that $m = 2n$ with some $n \in \mathbb{N}$. Because of (2.3), then $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q, 2n}^{\geq, \text{cd}}$, which, in view of (2.2), implies $L_n = 0_{q \times q}$ and that the block Hankel matrices H_n and H_{n-1} are non-negative Hermitian. Obviously,

$$H_n = \begin{bmatrix} H_{n-1} & y_{n, 2n-1} \\ z_{n, 2n-1} & s_{2n} \end{bmatrix} \quad \text{and} \quad H_n = \begin{bmatrix} s_0 & z_{1, n} \\ y_{1, n} & G_{n-1} \end{bmatrix}.$$

In the case $n \geq 2$, the matrix H_{n-1} admits the block representation

$$H_{n-1} = \begin{bmatrix} s_0 & z_{1, n-1} \\ y_{1, n-1} & G_{n-2} \end{bmatrix}.$$

Consequently, from [11, Lemmas 1.1.9 and 1.1.7] we get $\text{rank } H_n = \text{rank } H_{n-1} + \text{rank } L_n = \text{rank } H_{n-1}$ and $\text{rank } H_n = \text{rank } s_0 + \text{rank } \mathbb{L}_n$. Furthermore, $\text{rank } H_{n-1} = \text{rank } s_0$ in the case $n = 1$ and, in the case $n \geq 2$, moreover $\text{rank } H_{n-1} = \text{rank } s_0 + \text{rank } \mathbb{L}_{n-1}$. In view of (6.15), the last equations imply $\text{rank}(\mathbb{L}_n - \Xi_{n-1, 2n}) = 0$ in the case $n = 1$ and, in the case $n \geq 2$, furthermore $\text{rank}(\mathbb{L}_n - \Xi_{n-1, 2n}) = \text{rank}(\mathbb{L}_{n-1} - \Xi_{n-2, 2n})$. Using Lemma 6.11(b), we then conclude that $\text{rank}(-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)}) = 0$ in the case $n = 1$ and, in the case $n \geq 2$, that $\text{rank}(-\alpha H_{n-1}^{(1, \alpha)} + K_{n-1}^{(1, \alpha)}) = \text{rank}(-\alpha H_{n-2}^{(1, \alpha)} + K_{n-2}^{(1, \alpha)})$. Because of Remark 2.1, we have $(s_j^{(1, \alpha)})_{j=0}^{2n-1} \in \mathcal{K}_{q, 2n-1, \alpha}^{\geq}$, which, in view of (1.3) and (1.5), implies, that the sequence

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$(t_j)_{j=0}^{2n-2}$ defined by $t_j := -\alpha s_j^{(1,\alpha)} + s_{j+1}^{(1,\alpha)}$ for all $j \in \mathbb{Z}_{0,2n-2}$ belongs to $\mathcal{H}_{q,2n-2}^{\geq}$. Then $\text{rank } L_{n-1}^{(t)} = 0$ follows immediately in the case of $n = 1$ and in the case $n \geq 2$ from [11, Lemmas 1.1.9 and 1.1.7]. Thus, we have $L_{n-1}^{(t)} = 0_{q \times q}$. According to (2.2), we obtain then $(t_j)_{j=0}^{2n-2} \in \mathcal{H}_{q,2n-2}^{\geq, \text{cd}}$ and, in view of (2.4), consequently $(s_j^{(1,\alpha)})_{j=0}^{2n-1} \in \mathcal{K}_{q,2n-1,\alpha}^{\geq, \text{cd}}$. Thus, we have proved that $(s_j^{(1,\alpha)})_{j=0}^{\max\{0,m-1\}}$ belongs to $\mathcal{K}_{q,\max\{0,m-1\},\alpha}^{\geq}$, which, in view of (2.5), then implies $(s_j^{(1,\alpha)})_{j=0}^{\kappa-1} \in \mathcal{K}_{q,\kappa-1,\alpha}^{\geq, \text{cd}, \max\{0,m-1\}}$.

(e) Let $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq, \text{cd}}$. We first consider the case $\kappa \in \mathbb{N}$. In view of (2.3), (2.4), and (2.5), we have then $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq, \text{cd}, \kappa}$, which, according to (d), implies $(s_j^{(1,\alpha)})_{j=0}^{\kappa-1} \in \mathcal{K}_{q,\kappa-1,\alpha}^{\geq, \text{cd}, \kappa-1}$. Because of (2.5), hence $(s_j^{(1,\alpha)})_{j=0}^{\kappa-1} \in \mathcal{K}_{q,\kappa-1,\alpha}^{\geq, \text{cd}}$. Now we consider the case $\kappa = +\infty$. In view of (2.6), then $(s_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\infty,\alpha}^{\geq, \text{cd}, m}$ for some $m \in \mathbb{N}_0$, which, according to (d), implies $(s_j^{(1,\alpha)})_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\geq, \text{cd}, \max\{0,m-1\}}$. Because of (2.6), hence $(s_j^{(1,\alpha)})_{j=0}^{\infty} \in \mathcal{K}_{q,\infty,\alpha}^{\geq, \text{cd}}$. \square

7. The first α -Schur-transform of a sequence of complex matrices

This section plays a similar role as [25, Section 8]. Guided by our experiences from [25], we will choose a convenient two-sided normalization of the sequence introduced in (6.1). This construction gives us the tool to realize the elementary step in the Schur-type algorithm we are striving for. The following notion is one of the central objects in this paper. It describes the basic step of the Schur-type algorithm which will be developed in Section 8.

Definition 7.1. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Let the sequence $(s_j^{(1,\alpha)})_{j=0}^{\kappa-1}$ be given by (6.1). Then the sequence $(s_j^{[1,\alpha]})_{j=0}^{\kappa-1}$ defined by

$$s_j^{[1,\alpha]} := s_0 s_j^{(1,\alpha)} s_0 \quad \text{for all } j \in \mathbb{Z}_{0,\kappa-1}$$

is called the *first α -Schur-transform* (or short *α -S-transform*) of $(s_j)_{j=0}^{\kappa}$.

Our next considerations are aimed at studying the arithmetics of the first α -S-transform.

Remark 7.2. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence from $\mathbb{C}^{p \times q}$. Denote by $(s_j^{[1,\alpha]})_{j=0}^{\kappa-1}$ the first α -S-transform of $(s_j)_{j=0}^{\kappa}$. In view of Definition 7.1 and Remarks 6.2 and A.1, then $s_j^{(1,\alpha)} = s_0^\dagger s_j^{[1,\alpha]} s_0^\dagger$ for all $j \in \mathbb{Z}_{0,\kappa-1}$.

Remark 7.3. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. Denote by $(s_j^{[1,\alpha]})_{j=0}^{\kappa-1}$ the first α -S-transform of $(s_j)_{j=0}^{\kappa}$. In view of Remark 6.1, one can easily see that, for all $m \in \mathbb{Z}_{1,\kappa}$, the sequence $(s_j^{[1,\alpha]})_{j=0}^{m-1}$ depends only on the matrices s_0, s_1, \dots, s_m and, hence, it coincides with the α -S-transform of $(s_j)_{j=0}^m$.

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Remark 7.4. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. In view of Definition 7.1, for all $j \in \mathbb{Z}_{0,\kappa-1}$, then $\bigcup_{j=0}^{\kappa-1} \mathcal{R}(s_j^{[1,\alpha]}) \subseteq \mathcal{R}(s_0)$ and $\mathcal{N}(s_0) \subseteq \bigcap_{j=0}^{\kappa-1} \mathcal{N}(s_j^{[1,\alpha]})$.

Lemma 7.5. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then:*

- (a) *If $\gamma \in \mathbb{C}$, then $((\gamma s_j)^{[1,\alpha]})_{j=0}^{\kappa-1} = (\gamma s_j^{[1,\alpha]})_{j=0}^{\kappa-1}$ and $((\gamma^j s_j)^{[1,\alpha]})_{j=0}^{\kappa-1} = (\gamma^{j+1} s_j^{[1,\alpha]})_{j=0}^{\kappa-1}$*
- (b) *If $m \in \mathbb{N}$ and $L \in \mathbb{C}^{m \times p}$ with $\mathcal{R}(L^*) = \mathcal{R}(s_0)$, then $((L s_j)^{[1,\alpha]})_{j=0}^{\kappa-1} = (L(s_j^{[1,\alpha]}))_{j=0}^{\kappa-1}$.*
- (c) *If $n \in \mathbb{N}$ and $R \in \mathbb{C}^{q \times n}$ with $\mathcal{R}(s_0^*) = \mathcal{R}(R)$, then $((s_j R)^{[1,\alpha]})_{j=0}^{\kappa-1} = (s_j^{[1,\alpha]} R)_{j=0}^{\kappa-1}$.*
- (d) *If $m, n \in \mathbb{N}$, $L \in \mathbb{C}^{m \times p}$ with $\mathcal{R}(L^*) = \mathcal{R}(s_0)$ and $R \in \mathbb{C}^{q \times n}$ with $\mathcal{R}(s_0^*) = \mathcal{R}(R)$, then $((L s_j R)^{[1,\alpha]})_{j=0}^{\kappa-1} = (L s_j^{[1,\alpha]} R)_{j=0}^{\kappa-1}$.*
- (e) *If $m, n \in \mathbb{N}$, $U \in \mathbb{C}^{m \times p}$ with $U^* U = I_p$, and $V \in \mathbb{C}^{q \times n}$ with $V V^* = I_q$, then $((U s_j V)^{[1,\alpha]})_{j=0}^{\kappa-1} = (U s_j^{[1,\alpha]} V)_{j=0}^{\kappa-1}$.*
- (f) *If the sequence $(t_j)_{j=0}^\kappa$ is given by $t_j := s_j^*$ for all $j \in \mathbb{Z}_{0,\kappa}$, then $(s_j^{[1,\alpha]})^* = t_j^{[1,\bar{\alpha}]}$ for all $j \in \mathbb{Z}_{0,\kappa-1}$.*

Proof. (a) Use Remark 6.3(a).

(b) Because of $\mathcal{R}(L^*) = \mathcal{R}(s_0)$ we have $L^\dagger L s_0 = s_0$. Using Remark 6.3(b) for all $j \in \mathbb{Z}_{0,\kappa-1}$, we get then

$$(L s_j)^{[1,\alpha]} = (L s_0)(L s_j)^{(1,\alpha)}(L s_0) = L s_0 s_j^{(1,\alpha)} L^\dagger L s_0 = L s_0 s_j^{(1,\alpha)} s_0 = L s_j^{[1,\alpha]}.$$

(c) and (d) can be proved like (b) using parts (c) and (d) of Remark 6.3.

(e) and (f) can be proved using parts (e) and (f) of Remark 6.3. \square

Remark 7.6. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $n \in \mathbb{N}$. For all $m \in \mathbb{Z}_{1,n}$, let $p_m, q_m \in \mathbb{N}$ and let $(s_j^{(m)})_{j=0}^\kappa$ be a sequence of complex $p_m \times q_m$ matrices with first α -S-transform $(t_j^{(m)})_{j=0}^{\kappa-1}$. Then Remark 6.4 shows that $(\text{diag}[t_j^{(m)}]_{m=1}^n)_{j=0}^{\kappa-1}$ is exactly the first α -S-transform of $(\text{diag}[s_j^{(m)}]_{m=1}^n)_{j=0}^\kappa$.

Remark 7.7. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Further, let $(s_j^{[1,\alpha]})_{j=0}^{\kappa-1}$ be the first α -S-transform of $(s_j)_{j=0}^\kappa$. In view of Remark 6.5, for all $j \in \mathbb{Z}_{0,\kappa-1}$, then

$$s_j^{[1,\alpha]} = s_0 s_0^\dagger \sum_{l=0}^j s_{j+1-l}^{[+, \alpha]} s_l^{[\sharp, \alpha]} s_0 \quad \text{and} \quad s_j^{[1,\alpha]} = - \sum_{l=0}^{j+1} \alpha^{j+1-l} s_0 s_l^\dagger s_0.$$

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Lemma 7.8. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. For all $j \in \mathbb{Z}_{1,\kappa-1}$,

$$s_0^{[1,\alpha]} = s_0 s_0^\dagger s_1^{[+, \alpha]} s_0^\dagger s_0 \quad \text{and} \quad s_j^{[1,\alpha]} = s_0 s_0^\dagger \left(s_{j+1}^{[+, \alpha]} s_0^\dagger s_0 - \sum_{l=0}^{j-1} s_{j-l}^{[+, \alpha]} s_0^\dagger s_l^{[1,\alpha]} \right).$$

Proof. In view of Remarks 7.2 and 6.6, we get $s_0^{[1,\alpha]} = s_0 s_0^{(1,\alpha)} s_0 = s_0 s_0^\dagger s_1^{[+, \alpha]} s_0^\dagger s_0$ and, in the case $\kappa \geq 2$, for all $j \in \mathbb{Z}_{1,\kappa-1}$, furthermore

$$\begin{aligned} s_j^{[1,\alpha]} &= s_0 s_j^{(1,\alpha)} s_0 = s_0 \left(s_0^\dagger s_{j+1}^{[+, \alpha]} s_0^\dagger - s_0^\dagger \sum_{l=0}^{j-1} s_{j-l}^{[+, \alpha]} s_l^{(1,\alpha)} \right) s_0 \\ &= s_0 s_0^\dagger \left(s_{j+1}^{[+, \alpha]} s_0^\dagger s_0 - \sum_{l=0}^{j-1} s_{j-l}^{[+, \alpha]} s_l^{(1,\alpha)} s_0 \right) = s_0 s_0^\dagger \left(s_{j+1}^{[+, \alpha]} s_0^\dagger s_0 - \sum_{l=0}^{j-1} s_{j-l}^{[+, \alpha]} s_0^\dagger s_0 s_l^{(1,\alpha)} s_0 \right) \\ &= s_0 s_0^\dagger \left(s_{j+1}^{[+, \alpha]} s_0^\dagger s_0 - \sum_{l=0}^{j-1} s_{j-l}^{[+, \alpha]} s_0^\dagger s_l^{[1,\alpha]} \right). \quad \square \end{aligned}$$

Lemma 7.9. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa \in \mathcal{D}_{p \times q, \kappa}$. Then $s_0^{[1,\alpha]} = s_1^{[+, \alpha]}$ and, for all $j \in \mathbb{Z}_{1,\kappa-1}$, furthermore $s_j^{[1,\alpha]} = s_{j+1}^{[+, \alpha]} - \sum_{l=0}^{j-1} s_{j-l}^{[+, \alpha]} s_0^\dagger s_l^{[1,\alpha]}$.

Proof. Remark 4.3(a) yields $(s_j^{[+, \alpha]})_{j=0}^\kappa \in \mathcal{D}_{p \times q, \kappa}$. Hence, in view of Lemma 7.8, (4.1), Definition 3.3, and parts (c) and (b) of Remark A.1, we get

$$s_0^{[1,\alpha]} = s_0 s_0^\dagger s_1^{[+, \alpha]} s_0^\dagger s_0 = s_0^{[+, \alpha]} (s_0^{[+, \alpha]})^\dagger s_1^{[+, \alpha]} (s_0^{[+, \alpha]})^\dagger s_0^{[+, \alpha]} = s_1^{[+, \alpha]}$$

and, in the case $\kappa \geq 2$, for all $j \in \mathbb{Z}_{1,\kappa-1}$, furthermore

$$\begin{aligned} s_j^{[1,\alpha]} &= s_0 s_0^\dagger s_{j+1}^{[+, \alpha]} s_0^\dagger s_0 - \sum_{l=0}^{j-1} s_0 s_0^\dagger s_{j-l}^{[+, \alpha]} s_0^\dagger s_l^{[1,\alpha]} \\ &= s_0^{[+, \alpha]} (s_0^{[+, \alpha]})^\dagger s_{j+1}^{[+, \alpha]} (s_0^{[+, \alpha]})^\dagger s_0^{[+, \alpha]} - \sum_{l=0}^{j-1} s_0^{[+, \alpha]} (s_0^{[+, \alpha]})^\dagger s_{j-l}^{[+, \alpha]} s_0^\dagger s_l^{[1,\alpha]} \\ &= s_{j+1}^{[+, \alpha]} - \sum_{l=0}^{j-1} s_{j-l}^{[+, \alpha]} s_0^\dagger s_l^{[1,\alpha]}. \quad \square \end{aligned}$$

Remark 7.10. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. In view of Lemma 6.7 and Remarks 6.6 and 7.2, for all $j \in \mathbb{Z}_{1,\kappa}$, then

$$s_0^\dagger s_j^{[+, \alpha]} s_0^\dagger s_0 = s_0^\dagger \sum_{l=0}^{j-1} s_{j-1-l}^{[+, \alpha]} s_0^\dagger s_l^{[1,\alpha]}.$$

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Lemma 7.11. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa \in \mathcal{D}_{p \times q, \kappa}$. For all $j \in \mathbb{Z}_{1, \kappa}$, then*

$$s_j^{[+, \alpha]} = \sum_{l=0}^{j-1} s_{j-1-l}^{[+, \alpha]} s_0^\dagger s_l^{[1, \alpha]}.$$

Proof. Remark 4.3(a) yields $(s_j^{[+, \alpha]})_{j=0}^\kappa \in \mathcal{D}_{p \times q, \kappa}$. Hence, in view of Definition 3.3, parts (c) and (b) of Remark A.1, (4.1), and Remark 7.10, for all $j \in \mathbb{Z}_{1, \kappa}$, we get

$$\begin{aligned} s_j^{[+, \alpha]} &= s_0^{[+, \alpha]} (s_0^{[+, \alpha]})^\dagger s_j^{[+, \alpha]} (s_0^{[+, \alpha]})^\dagger s_0^{[+, \alpha]} = s_0^{[+, \alpha]} s_0^\dagger s_j^{[+, \alpha]} s_0^\dagger s_0 \\ &= s_0^{[+, \alpha]} s_0^\dagger \sum_{l=0}^{j-1} s_{j-1-l}^{[+, \alpha]} s_0^\dagger s_l^{[1, \alpha]} = \sum_{l=0}^{j-1} s_0^{[+, \alpha]} (s_0^{[+, \alpha]})^\dagger s_{j-1-l}^{[+, \alpha]} s_0^\dagger s_l^{[1, \alpha]} = \sum_{l=0}^{j-1} s_{j-1-l}^{[+, \alpha]} s_0^\dagger s_l^{[1, \alpha]}. \end{aligned}$$

□

Lemma 7.12. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ and $(t_j)_{j=0}^\kappa$ be sequences of complex $p \times q$ matrices. Then the following statements are equivalent:*

(i) $s_j^{[1, \alpha]} = t_j^{[1, \alpha]}$ for all $j \in \mathbb{Z}_{0, \kappa-1}$ and $s_0 = t_0$.

(ii) $s_0 s_0^\dagger s_j s_0^\dagger s_0 = t_0 t_0^\dagger t_j t_0^\dagger t_0$ for all $j \in \mathbb{Z}_{0, \kappa}$.

Proof. According to Remark 7.2, statement (i) is equivalent to

(iii) $s_j^{(1, \alpha)} = t_j^{(1, \alpha)}$ for all $j \in \mathbb{Z}_{0, \kappa-1}$ and $s_0 = t_0$.

which, in view of Lemma 6.8, is equivalent to (ii). □

For all $m \in \mathbb{N}_0$ and all $w \in \mathbb{C}$, we easily see, in view of the block structure given in (4.3), that

$$(I_{m+1} \otimes s_0) [R_{q, m}(w)] = [R_{p, m}(w)] (I_{m+1} \otimes s_0) \quad (7.1)$$

and

$$[R_{p, m}(w)]^* (I_{m+1} \otimes s_0) = (I_{m+1} \otimes s_0) [R_{q, m}(w)]^*. \quad (7.2)$$

Remark 7.13. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, let $(s_j)_{j=0}^\kappa$ be a sequence from $\mathbb{C}^{p \times q}$, and let $n \in \mathbb{Z}_{0, \kappa-1}$. In view of (3.1), Remark 6.9, (5.1), (7.1), and (7.2), then

$$\begin{aligned} \mathbf{S}_n^{(s^{[1, \alpha]})} &= -\nabla_{p, n+1}^* \left([R_{p, n+1}(\alpha)] (I_{n+2} \otimes s_0) \mathbf{S}_{n+1}^\# (I_{n+2} \otimes s_0) - (I_{n+2} \otimes s_0) \right) \Delta_{q, n+1} \\ &= (T_{1, n}^* \otimes s_0) - \nabla_{p, n+1}^* [R_{p, n+1}(\alpha)] (I_{n+2} \otimes s_0) \mathbf{S}_{n+1}^\# (I_{n+2} \otimes s_0) \Delta_{q, n+1} \end{aligned}$$

and

$$\begin{aligned} \mathbb{S}_n^{(s^{[1, \alpha]})} &= -\Delta_{p, n+1}^* \left((I_{n+2} \otimes s_0) \mathbf{S}_{n+1}^\# (I_{n+2} \otimes s_0) [R_{q, n+1}(\bar{\alpha})]^* - (I_{n+2} \otimes s_0) \right) \nabla_{q, n+1} \\ &= (T_{1, n} \otimes s_0) - \Delta_{p, n+1}^* (I_{n+2} \otimes s_0) \mathbf{S}_{n+1}^\# (I_{n+2} \otimes s_0) [R_{q, n+1}(\bar{\alpha})]^* \nabla_{q, n+1}. \end{aligned}$$

Notation 7.14. For all $n \in \mathbb{N}_0$, let $\mathfrak{L}_{q, n}$ (resp. $\mathfrak{U}_{q, n}$) be the set of all complex $(n+1)q \times (n+1)q$ matrices A which satisfy the following condition:

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- If $A = [A_{jk}]_{j,k=0}^n$ is the $q \times q$ block representation of A , then $A_{jj} = I_q$ for all $j \in \mathbb{Z}_{0,n}$ and $A_{jk} = 0_{q \times q}$ for all $j, k \in \mathbb{Z}_{0,n}$ with $j < k$ (resp. $k < j$).

Remark 7.15. Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. In view of (3.2), Definition 3.1, and Notation 7.14, then, for all $m \in \mathbb{Z}_{0,\kappa}$, the matrix

$$\mathbf{D}_m^{(s)} := (I_{m+1} \otimes s_0) \mathbf{S}_m^\# + \left[I_{m+1} \otimes (I_p - s_0 s_0^\dagger) \right] \quad (7.3)$$

belongs to $\mathfrak{L}_{p,m}$, the matrix

$$\mathbb{D}_m^{(s)} := \mathbb{S}_m^\# (I_{m+1} \otimes s_0) + \left[I_{m+1} \otimes (I_q - s_0^\dagger s_0) \right] \quad (7.4)$$

belongs to $\mathfrak{U}_{q,m}$, and, according to Remark A.2, in particular, $\det \mathbf{D}_m^{(s)} = 1$ and $\det \mathbb{D}_m^{(s)} = 1$.

For short, we will also write \mathbf{D}_m and \mathbb{D}_m for $\mathbf{D}_m^{(s)}$ and $\mathbb{D}_m^{(s)}$, respectively.

Lemma 7.16. *Let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$ and let $(s_j)_{j=0}^\kappa$ be a sequence of Hermitian complex $q \times q$ matrices. For all $m \in \mathbb{Z}_{0,\kappa}$, then $\mathbf{D}_m^* = \mathbb{D}_m$.*

Proof. Let $m \in \mathbb{Z}_{0,\kappa}$. From [26, Corollary 5.17] we obtain then $(s_j^\#)^* = s_j^\#$ for all $j \in \mathbb{Z}_{0,m}$ which, in view of (3.2) and (3.1), implies $(\mathbf{S}_m^\#)^* = \mathbb{S}_m^\#$. Using Remark A.1(a), we get furthermore $(s_0 s_0^\dagger)^* = (s_0^\dagger)^* s_0^* = (s_0^*)^\dagger s_0^* = s_0^\dagger s_0$. Taking additionally into account (7.3) and (7.4), we finally obtain the asserted equation. \square

Remark 7.17. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices with $[+, \alpha]$ -transform $(t_j)_{j=0}^\kappa$. In view of (7.3), (7.4), (4.1), Remark 4.7, (7.1), and (7.2), then one can easily see that, for all $m \in \mathbb{Z}_{0,\kappa}$, the matrices

$$\mathbf{D}_m^{[+, \alpha]} := \mathbf{D}_m^{(t)} \quad \text{and} \quad \mathbb{D}_m^{[+, \alpha]} := \mathbb{D}_m^{(t)} \quad (7.5)$$

can be represented via

$$\mathbf{D}_m^{[+, \alpha]} = [R_{p,m}(\alpha)] (I_{m+1} \otimes s_0) \mathbf{S}_m^\# + \left[I_{m+1} \otimes (I_p - s_0 s_0^\dagger) \right]$$

and

$$\mathbb{D}_m^{[+, \alpha]} = \mathbb{S}_m^\# (I_{m+1} \otimes s_0) [R_{q,m}(\overline{\alpha})]^* + \left[I_{m+1} \otimes (I_q - s_0^\dagger s_0) \right].$$

We now turn our attention to several block Hankel matrices built from the first α -S-transform. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then, let

$$H_n^{[1, \alpha]} := [s_{j+k}^{[1, \alpha]}]_{j,k=0}^n \quad (7.6)$$

for all $n \in \mathbb{N}_0$ with $2n \leq \kappa - 1$ and let

$$K_n^{[1, \alpha]} := [s_{j+k+1}^{[1, \alpha]}]_{j,k=0}^n \quad (7.7)$$

for all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa - 1$.

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Remark 7.18. Let $\alpha \in \mathbb{C}$, let $n \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $p \times q$ matrices. Then because of (7.6), Definition 7.1, and (6.2), we have $H_n^{[1,\alpha]} = (I_{n+1} \otimes s_0)H_n^{(1,\alpha)}(I_{n+1} \otimes s_0)$.

The following result contains interesting links between the block Hankel matrices $H_n^{[1,\alpha]}$ and $H_{\alpha \triangleright n}$ introduced via (7.6) and (2.15), respectively.

Lemma 7.19. *Let $\alpha \in \mathbb{C}$, let $n \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{2n+1} \in \tilde{\mathcal{D}}_{p \times q, 2n+1}$. Let the matrices $\Xi_{n, 2n+1}$, $\mathbf{D}_n^{[+, \alpha]}$, and $\mathbb{D}_n^{[+, \alpha]}$ be given via (5.4), (7.5), (7.3), and (7.4), respectively. Then:*

(a)

$$H_n^{[1,\alpha]} = [R_{p,n}(\alpha)] (I_{n+1} \otimes s_0) \mathbf{S}_n^\dagger H_{\alpha \triangleright n} \mathbb{S}_n^\dagger (I_{n+1} \otimes s_0) [R_{q,n}(\bar{\alpha})]^* \quad (7.8)$$

and

$$H_n^{[1,\alpha]} = \mathbf{D}_n^{[+, \alpha]} (H_{\alpha \triangleright n} - \Xi_{n, 2n+1}) \mathbb{D}_n^{[+, \alpha]}. \quad (7.9)$$

(b) $\text{rank}(H_n^{[1,\alpha]}) = \text{rank}(H_{\alpha \triangleright n} - \Xi_{n, 2n+1})$.

(c) If $p = q$, then $\det(H_n^{[1,\alpha]}) = (\det s_0)(\det s_0)^\dagger \det H_{\alpha \triangleright n}$ and $\det(H_n^{[1,\alpha]}) = \det(H_{\alpha \triangleright n} - \Xi_{n, 2n+1})$.

Proof. (a) Using Remark 7.18, Lemma 6.10(a), (7.1), and (7.2), then (7.8) follows. Since $(s_j)_{j=0}^{2n+1}$ belongs to $\tilde{\mathcal{D}}_{p \times q, 2n+1}$, Definition 3.7 yields $(s_j)_{j=0}^{2n} \in \mathcal{D}_{p \times q, 2n}$. Hence, according to Proposition 3.5, we have

$$\mathbf{S}_k^\sharp = \mathbf{S}_k^\dagger \quad \text{and} \quad \mathbb{S}_k^\sharp = \mathbb{S}_k^\dagger \quad (7.10)$$

for all $k \in \mathbb{Z}_{0, 2n}$. From Definition 3.3 and Remark A.1 we also see that

$$s_0 s_0^\dagger s_j = s_j \quad \text{and} \quad s_j s_0^\dagger s_0 = s_j \quad (7.11)$$

hold true for all $j \in \mathbb{Z}_{0, 2n}$. In view of (2.15), (2.8), (1.3), and (5.4), the equations in (7.11) immediately imply

$$\left[I_{n+1} \otimes (I_p - s_0 s_0^\dagger) \right] (H_{\alpha \triangleright n} - \Xi_{n, 2n+1}) = 0_{(n+1)p \times (n+1)q}$$

and

$$(H_{\alpha \triangleright n} - \Xi_{n, 2n+1}) \left[I_{n+1} \otimes (I_q - s_0^\dagger s_0) \right] = 0_{(n+1)p \times (n+1)q}.$$

Consequently, Remark 7.17 yields

$$\begin{aligned} & \mathbf{D}_n^{[+, \alpha]} (H_{\alpha \triangleright n} - \Xi_{n, 2n+1}) \mathbb{D}_n^{[+, \alpha]} \\ &= [R_{p,n}(\alpha)] (I_{n+1} \otimes s_0) \mathbf{S}_n^\sharp (H_{\alpha \triangleright n} - \Xi_{n, 2n+1}) \mathbb{S}_n^\sharp (I_{n+1} \otimes s_0) [R_{q,n}(\bar{\alpha})]^*. \end{aligned} \quad (7.12)$$

Taking (3.2), (3.1), Definition 3.1, and (5.4) into account, we get $\mathbf{S}_n^\sharp \Xi_{n, 2n+1} \mathbb{S}_n^\sharp = 0_{(n+1)q \times (n+1)p}$, whereas (7.10) for $k = n$ shows that $\mathbf{S}_n^\sharp = \mathbf{S}_n^\dagger$ and $\mathbb{S}_n^\sharp = \mathbb{S}_n^\dagger$ hold true.

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Thus, the expression on the right-hand side of (7.12) coincides with the expression on the right-hand side of (7.8). Hence, (7.9) is proved as well.

(b) This follows from (7.9), (7.5), and Remark 7.15.

(c) From (4.3) we know that $\det[R_{p,n}(\alpha)] = 1 = \det([R_{q,n}(\bar{\alpha})]^*)$. Equations (7.10) and (3.2) and Definition 3.1 yield

$$\det[(I_{n+1} \otimes s_0)\mathbf{S}_n^\dagger] = \det(I_{n+1} \otimes s_0) \det(\mathbf{S}_n^\#) = (\det s_0)^{n+1} [\det(s_0^\dagger)]^{n+1}$$

and, similarly

$$\det[\mathbf{S}_n^\dagger(I_{n+1} \otimes s_0)] = [\det(s_0^\dagger)]^{n+1} (\det s_0)^{n+1}.$$

Since

$$(\det s_0)^{2n+2} [\det(s_0^\dagger)]^{2n+2} = (\det s_0)(\det s_0)^\dagger$$

is true, equation (7.8) then implies the first equation stated in (c). Remark 7.15 and (7.5) show that $\det \mathbf{D}_n^{[+, \alpha]} = 1 = \det \mathbb{D}_n^{[+, \alpha]}$. Thus, (7.9) shows that the second equation stated in part (c) is also true. \square

Lemma 7.20. *Let $\alpha \in \mathbb{C}$, let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n} \in \tilde{\mathcal{D}}_{p \times q, 2n}$. Let the matrices \mathbb{L}_n , $\Xi_{n-1, 2n}$, \mathbf{D}_{n-1} , and \mathbb{D}_{n-1} be given via (2.12), (5.4), (7.3), and (7.4), respectively. Then:*

(a)

$$-\alpha H_{n-1}^{[1, \alpha]} + K_{n-1}^{[1, \alpha]} = (I_n \otimes s_0) \mathbf{S}_{n-1}^\dagger \mathbb{L}_n \mathbf{S}_{n-1}^\dagger (I_n \otimes s_0) \quad (7.13)$$

and

$$-\alpha H_{n-1}^{[1, \alpha]} + K_{n-1}^{[1, \alpha]} = \mathbf{D}_{n-1} (\mathbb{L}_n - \Xi_{n-1, 2n}) \mathbb{D}_{n-1}. \quad (7.14)$$

(b) $\text{rank}(-\alpha H_{n-1}^{[1, \alpha]} + K_{n-1}^{[1, \alpha]}) = \text{rank}(\mathbb{L}_n - \Xi_{n-1, 2n})$.

(c) If $p = q$, then

$$\det(-\alpha H_{n-1}^{[1, \alpha]} + K_{n-1}^{[1, \alpha]}) = (\det s_0)(\det s_0)^\dagger \det \mathbb{L}_n \quad (7.15)$$

and

$$\det(-\alpha H_{n-1}^{[1, \alpha]} + K_{n-1}^{[1, \alpha]}) = \det(\mathbb{L}_n - \Xi_{n-1, 2n}). \quad (7.16)$$

Proof. (a) Equation (7.13) immediately follows from (7.6), (7.7), Definition 7.1, (6.2), (6.3), and Lemma 6.11(a). Since $n \in \mathbb{N}$ and $(s_j)_{j=0}^{2n} \in \tilde{\mathcal{D}}_{p \times q, 2n}$ hold, we have (7.11) for all $j \in \mathbb{Z}_{0, 2n-1}$ and $(s_j)_{j=0}^n \in \mathcal{D}_{p \times q, n}$. Thus Proposition 3.5 yields (7.10) for all $k \in \mathbb{Z}_{0, n}$. From (3.2), (3.1), Definition 3.1, and (5.4) we also see that $\mathbf{S}_{n-1}^\# \Xi_{n-1, 2n} \mathbf{S}_n^\# = 0$ is true. Using (2.12), (2.10), (2.7), (5.4), and (7.11), we get $[I_n \otimes (I_p - s_0 s_0^\dagger)](\mathbb{L}_n - \Xi_{n-1, 2n}) = 0_{np \times nq}$ and $(\mathbb{L}_n - \Xi_{n-1, 2n})[I_n \otimes (I_q - s_0^\dagger s_0)] = 0_{np \times nq}$. Thus, because of (7.10), from (7.3), (7.4), and (7.13) then (7.14) follows.

(b) This follows from (7.14) and Remark 7.15.

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(c) Because of (7.10), (3.2), and Definition 3.1, we have

$$\begin{aligned} \det(I_n \otimes s_0) \det(\mathbf{S}_{n-1}^\dagger) \det(\mathbf{S}_{n-1}^\dagger) \det(I_n \otimes s_0) &= (\det s_0)^n \left[\det(s_0^\dagger) \right]^n \left[\det(s_0^\dagger) \right]^n (\det s_0)^n \\ &= (\det s_0)(\det s_0)^\dagger. \end{aligned}$$

Using this, from (7.13), we get (7.15). Equation (7.16) follows from (7.14) and Remark 7.15. \square

In [9, 27] Chen and Hu treat the truncated matricial Stieltjes moment problem ($\alpha = 0$ in our setting). They introduce a transformation Γ_k which maps a sequence of length $k + 1$ of complex square matrices to a sequence of length k of complex square matrices. This transformation is defined via [9, formula (9)] (see also [27, formula (3.2)]). A closer look on [9, formula (9)] shows that this identity is essentially (7.13) for $\alpha = 0$ and $p = q$. Therefore, the transformation Γ_m coincides for sequences from $\tilde{\mathcal{D}}_{q \times q, m}$ with the first 0-Schur-transformation. To describe the respective solution sets, Chen and Hu reduce the length of the given sequence of prescribed moments in each step by 2, using the transformation $\Gamma_{m-1}\Gamma_m$ (see [9, formula (12)] and [27, formula (3.7)]).

Now we state the main result of this section.

Theorem 7.21. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $q \times q$ matrices. Then:*

- (a) *If $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^\geq$, then $(s_j^{[1, \alpha]})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^\geq$.*
- (b) *If $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$, then $(s_j^{[1, \alpha]})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^{\geq, e}$.*
- (c) *If $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^>$, then $(s_j^{[1, \alpha]})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^>$.*
- (d) *If $m \in \mathbb{Z}_{0, \kappa}$ and $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, \text{cd}, m}$, then $(s_j^{[1, \alpha]})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^{\geq, \text{cd}, \max\{0, m-1\}}$.*
- (e) *If $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, \text{cd}}$, then $(s_j^{[1, \alpha]})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^{\geq, \text{cd}}$.*

Proof. (a) Let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^\geq$. In view of Lemma 2.3(a), then $s_0^* = s_0$. Thus, Definition 7.1, Proposition 6.13(a), and Remark 2.2 yield $(s_j^{[1, \alpha]})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^\geq$.

(b) Let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$. Then $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^\geq$. Lemma 2.3(a) shows that $s_0^* = s_0$. Thus, Definition 7.1, Proposition 6.13(b), and Remark 2.2 yield $(s_j^{[1, \alpha]})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^{\geq, e}$.

(c) Let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^>$. Then, in view of (1.1), the matrix s_0 is positive Hermitian and hence Hermitian and non-singular. Thus, Definition 7.1, Proposition 6.13(c), and [21, Remark 2.15] yield $(s_j^{[1, \alpha]})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^>$.

(d) Let $m \in \mathbb{Z}_{0, \kappa}$ and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, \text{cd}, m}$. Because of (2.5), then $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^\geq$. Hence, in view of (a), we have $(s_j^{[1, \alpha]})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^\geq$ and, because of Lemma 2.3(a), furthermore $s_0^* = s_0$. The application of Proposition 6.13(d) yields $(s_j^{(1, \alpha)})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^{\geq, \text{cd}, \max\{0, m-1\}}$ which, in view of (2.5), implies $(s_j^{(1, \alpha)})_{j=0}^{\max\{0, m-1\}} \in \mathcal{K}_{q, \max\{0, m-1\}, \alpha}^{\geq, \text{cd}}$.

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Using Remark 6.2 and $s_0^* = s_0$ we obtain $\mathcal{N}(s_0) \subseteq \bigcap_{j=0}^{\max\{0, m-1\}-1} \mathcal{N}(s_j^{(1, \alpha)})$ in the case $\max\{0, m-1\} \geq 1$. Thus, Definition 7.1 and [21, Lemma 5.7] yield $(s_j^{[1, \alpha]})_{j=0}^{\max\{0, m-1\}} \in \mathcal{K}_{q, \max\{0, m-1\}, \alpha}^{\geq, \text{cd}}$. Hence, in view of (2.5), we obtain $(s_j^{[1, \alpha]})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^{\geq, \text{cd}}$.

(e) Let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, \text{cd}}$. Then $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq}$. Hence, in view of Lemma 2.3(a), we have $s_0^* = s_0$. The application of Proposition 6.13(e) yields $(s_j^{(1, \alpha)})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^{\geq, \text{cd}}$. Using Remark 6.2 and $s_0^* = s_0$, we obtain $\mathcal{N}(s_0) \subseteq \bigcap_{j=0}^{\kappa-2} \mathcal{N}(s_j^{(1, \alpha)})$ in the case $\kappa-1 \geq 1$. Thus, Definition 7.1 and [21, Lemma 5.7] yield $(s_j^{[1, \alpha]})_{j=0}^{\kappa-1} \in \mathcal{K}_{q, \kappa-1, \alpha}^{\geq, \text{cd}}$. \square

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The α -Schur-transform introduced in Section 7 generates in a natural way a corresponding algorithm for (finite or infinite) sequences of complex $p \times q$ matrices. The investigation of this algorithm is the central point of this section. First we are going to extend Definition 7.1.

Definition 8.1. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. The sequence $(s_j^{[0, \alpha]})_{j=0}^\kappa$ given by $s_j^{[0, \alpha]} := s_j$ for all $j \in \mathbb{Z}_{0, \kappa}$ is called the 0-th α -S-transform of $(s_j)_{j=0}^\kappa$. In the case $\kappa \geq 1$, for all $k \in \mathbb{Z}_{1, \kappa}$, the k -th α -S-transform $(s_j^{[k, \alpha]})_{j=0}^{\kappa-k}$ of $(s_j)_{j=0}^\kappa$ is recursively defined by $s_j^{[k, \alpha]} := t_j^{[1, \alpha]}$ for all $j \in \mathbb{Z}_{0, \kappa-k}$, where $(t_j)_{j=0}^{\kappa-(k-1)}$ denotes the $(k-1)$ -th α -S-transform of $(s_j)_{j=0}^\kappa$.

Remark 8.2. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices, and let $k \in \mathbb{Z}_{0, \kappa}$. Denote by $(s_j^{[k, \alpha]})_{j=0}^{\kappa-k}$ the k -th α -S-transform of $(s_j)_{j=0}^\kappa$. From Definition 8.1 and Remark 7.3 one can see then that, for all $m \in \mathbb{Z}_{k, \kappa}$, the sequence $(s_j^{[k, \alpha]})_{j=0}^{m-k}$ depends only on the matrices s_0, s_1, \dots, s_m and, hence, it coincides with the k -th α -S-transform of $(s_j)_{j=0}^m$. In particular, the sequence $(s_j^{[k, \alpha]})_{j=0}^0$ depends only on the matrices s_0, s_1, \dots, s_k and is the k -th α -S-transform of $(s_j)_{j=0}^k$.

Remark 8.3. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. From Definition 8.1 then it is immediately obvious that, for all $k \in \mathbb{Z}_{0, \kappa}$ and all $l \in \mathbb{Z}_{0, \kappa-k}$, the $(k+l)$ -th α -S-transform $(s_j^{[k+l, \alpha]})_{j=0}^{\kappa-(k+l)}$ of $(s_j)_{j=0}^\kappa$ is exactly the l -th α -S-transform of the k -th α -S-transform $(s_j^{[k, \alpha]})_{j=0}^{\kappa-k}$ of $(s_j)_{j=0}^\kappa$.

Lemma 8.4. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices, and let $k \in \mathbb{Z}_{0, \kappa-1}$. Then $\bigcup_{l=1}^{\kappa-k} [\bigcup_{j=0}^{\kappa-(k+l)} \mathcal{R}(s_j^{[k+l, \alpha]})] \subseteq \mathcal{R}(s_0^{[k, \alpha]})$ and $\mathcal{N}(s_0^{[k, \alpha]}) \subseteq \bigcap_{l=1}^{\kappa-k} [\bigcap_{j=0}^{\kappa-(k+l)} \mathcal{N}(s_j^{[k+l, \alpha]})]$.

Proof. Let $l \in \mathbb{Z}_{1, \kappa-k}$. Then $\kappa - (k+l-1) \geq 1$. Hence, having in mind Definition 8.1,

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the application of Remark 7.4 to the sequence $(s_j^{[k+l-1, \alpha]})_{j=0}^{\kappa-(k+l-1)}$ yields

$$\bigcup_{j=0}^{\kappa-(k+l)} \mathcal{R}(s_j^{[k+l, \alpha]}) \subseteq \mathcal{R}(s_0^{[k+l-1, \alpha]}) \quad \text{and} \quad \mathcal{N}(s_0^{[k+l-1, \alpha]}) \subseteq \bigcap_{j=0}^{\kappa-(k+l)} \mathcal{N}(s_j^{[k+l, \alpha]}).$$

Thus, it is sufficient to show $\mathcal{R}(s_0^{[k+l-1, \alpha]}) \subseteq \mathcal{R}(s_0^{[k, \alpha]})$ and $\mathcal{N}(s_0^{[k, \alpha]}) \subseteq \mathcal{N}(s_0^{[k+l-1, \alpha]})$. However, keeping in mind that the case $l = 1$ is trivial, these inclusions follow by induction using Definition 8.1 and Remark 7.4. \square

Remark 8.5. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices. In view of Lemma 8.4, for all $k \in \mathbb{Z}_{0, \kappa-1}$, all $l \in \mathbb{Z}_{1, \kappa-k}$, and all $j \in \mathbb{Z}_{0, \kappa-(k+l)}$, then $\mathcal{R}(s_j^{[k+l, \alpha]}) \subseteq \mathcal{R}(s_0^{[k, \alpha]})$ and $\mathcal{N}(s_0^{[k, \alpha]}) \subseteq \mathcal{N}(s_j^{[k+l, \alpha]})$ and, in particular, $\text{rank}(s_j^{[k+l, \alpha]}) \leq \text{rank}(s_0^{[k, \alpha]})$.

Lemma 8.6. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $p \times q$ matrices, and let $k \in \mathbb{Z}_{0, \kappa}$. Then:*

- (a) *If $\gamma \in \mathbb{C}$, then $((\gamma s_j)^{[k, \alpha]})_{j=0}^{\kappa-k} = (\gamma s_j^{[k, \alpha]})_{j=0}^{\kappa-k}$ and $((\gamma^j s_j)^{[k, \alpha]})_{j=0}^{\kappa-k} = (\gamma^{j+k} s_j^{[k, \alpha]})_{j=0}^{\kappa-k}$.*
- (b) *If $m, n \in \mathbb{N}$, $U \in \mathbb{C}^{m \times p}$ with $U^* U = I_p$, and $V \in \mathbb{C}^{q \times n}$ with $V V^* = I_q$, then $((U s_j V)^{[k, \alpha]})_{j=0}^{\kappa-k} = (U s_j^{[k, \alpha]} V)_{j=0}^{\kappa-k}$.*
- (c) *If the sequence $(t_j)_{j=0}^{\kappa}$ is given by $t_j := s_j^*$ for all $j \in \mathbb{Z}_{0, \kappa}$, then $(s_j^{[k, \alpha]})^* = t_j^{[k, \bar{\alpha}]}$ for all $j \in \mathbb{Z}_{0, \kappa-k}$.*

Proof. Use Definition 8.1 and Lemma 7.5. \square

Remark 8.7. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(s_j)_{j=0}^{\kappa}$ be a sequence of Hermitian complex $q \times q$ matrices, and let $k \in \mathbb{Z}_{0, \kappa}$. In view of Lemma 8.6(c), then $(s_j^{[k, \alpha]})^* = s_j^{[k, \alpha]}$ for all $j \in \mathbb{Z}_{0, \kappa-k}$.

Remark 8.8. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $k \in \mathbb{Z}_{0, \kappa}$, and let $n \in \mathbb{N}$. For all $m \in \mathbb{Z}_{1, n}$, let $p_m, q_m \in \mathbb{N}$, and let $(s_j^{(m)})_{j=0}^{\kappa}$ be a sequence of complex $p_m \times q_m$ matrices with k -th α -S-transform $(t_j^{(m)})_{j=0}^{\kappa-k}$. In view of Definition 8.1 and Remark 7.6, then $(\text{diag}[t_j^{(m)}]_{m=1}^n)_{j=0}^{\kappa-k}$ is exactly the k -th α -S-transform of $(\text{diag}[s_j^{(m)}]_{m=1}^n)_{j=0}^{\kappa}$.

Remark 8.9. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, let $(s_j)_{j=0}^{\kappa}$ and $(t_j)_{j=0}^{\kappa}$ be sequences of complex $p \times q$ matrices such that $s_0 s_0^\dagger s_j s_0^\dagger s_0 = t_0 t_0^\dagger t_j t_0^\dagger t_0$ for all $j \in \mathbb{Z}_{0, \kappa}$, and let $k \in \mathbb{Z}_{1, \kappa}$. In view of Definition 8.1 and Lemma 7.12, then $s_j^{[k, \alpha]} = t_j^{[k, \alpha]}$ for all $j \in \mathbb{Z}_{0, \kappa-k}$.

Now we are going to study the Schur-type algorithm introduced in Definition 8.1 for sequences belonging to the class $\mathcal{K}_{q, \kappa, \alpha}^{\geq}$ and its distinguished subclasses.

Now we state the main result of this section.

Theorem 8.10. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(s_j)_{j=0}^{\kappa}$ be a sequence of complex $q \times q$ matrices, and let $k \in \mathbb{Z}_{0, \kappa}$. Then:*

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- (a) If $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq}$, then $(s_j^{[k,\alpha]})_{j=0}^{\kappa-k} \in \mathcal{K}_{q,\kappa-k,\alpha}^{\geq}$.
- (b) If $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$, then $(s_j^{[k,\alpha]})_{j=0}^{\kappa-k} \in \mathcal{K}_{q,\kappa-k,\alpha}^{\geq,e}$.
- (c) If $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{>}$, then $(s_j^{[k,\alpha]})_{j=0}^{\kappa-k} \in \mathcal{K}_{q,\kappa-k,\alpha}^{>}$.
- (d) If $m \in \mathbb{Z}_{0,\kappa}$ and $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,\text{cd},m}$, then $(s_j^{[k,\alpha]})_{j=0}^{\kappa-k} \in \mathcal{K}_{q,\kappa-k,\alpha}^{\geq,\text{cd},\max\{0,m-k\}}$.
- (e) If $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,\text{cd}}$, then $(s_j^{[k,\alpha]})_{j=0}^{\kappa-k} \in \mathcal{K}_{q,\kappa-k,\alpha}^{\geq,\text{cd}}$.

Proof. In view of Definition 8.1, the case $k = 0$ is trivial. Thus, there is an $l \in \mathbb{Z}_{0,\kappa}$ such that the stated implications hold true for all $k \in \mathbb{Z}_{0,l}$. If $l = \kappa$, then the proof is complete. Assume that $l < \kappa$. Then Definition 8.1 and Theorem 7.21 show that the stated implications also hold true for $k = l + 1$. Thus, the assertion inductively follows. \square

Remark 8.11. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq}$. In view of Theorem 8.10(a) and Lemma 2.3(b), then $s_0^{[k,\alpha]} \in \mathbb{C}_{\geq}^{q \times q}$ for all $k \in \mathbb{Z}_{0,\kappa}$.

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In this section, we mainly concentrate our considerations to the class $\mathcal{K}_{q,\kappa,\alpha}^{\geq}$ or one of its distinguished subclasses $\mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$, $\mathcal{K}_{q,\kappa,\alpha}^{>}$, and $\mathcal{K}_{q,\kappa,\alpha}^{\geq,\text{cd}}$. In this situation, we will recognize that the right α -Stieltjes parametrization of $(s_j)_{j=0}^\kappa$ introduced in Definition 2.8 is generated by the Schur-type algorithm applied to $(s_j)_{j=0}^\kappa$. The following considerations are aimed at establishing block *LDU* decompositions of several block Hankel matrices. In order to realize this goal, we first derive some matrix identities.

Lemma 9.1. *Let $\alpha \in \mathbb{C}$, let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n} \in \tilde{\mathcal{D}}_{p \times q, 2n}$. Let \mathbf{D}_n and \mathbb{D}_n be given via (7.3) and (7.4), respectively, and let $\Xi_{n,2n}$ be given via (5.4). Then*

$$\mathbf{D}_n H_n \mathbb{D}_n = \text{diag}[s_0, -\alpha H_{n-1}^{[1,\alpha]} + K_{n-1}^{[1,\alpha]}] + \Xi_{n,2n}. \quad (9.1)$$

Proof. From Lemma 6.12 we get (6.12). In view of Definition 3.7, we have $(s_j)_{j=0}^{2n-1} \in \mathcal{D}_{p \times q, 2n-1}$. According to Proposition 3.5, thus $\mathbf{S}_n^\dagger = \mathbf{S}_n^\#$ and $\mathbb{S}_n^\dagger = \mathbb{S}_n^\#$. Furthermore, $s_0^\# = s_0^\dagger$ by Definition 3.1. Taking additionally into account (6.12), (6.2), (6.3), Definition 7.1, (7.6), and (7.7) we obtain

$$(I_{n+1} \otimes s_0) \mathbf{S}_n^\# H_n \mathbb{S}_n^\# (I_{n+1} \otimes s_0) = \text{diag}[s_0, -\alpha H_{n-1}^{[1,\alpha]} + K_{n-1}^{[1,\alpha]}].$$

In view of $(s_j)_{j=0}^{2n-1} \in \mathcal{D}_{p \times q, 2n-1}$, we get from Definition 3.3 and parts (c) and (b) of Remark A.1 furthermore $s_0 s_0^\dagger s_j = s_j$ and $s_j s_0^\dagger s_0 = s_j$ for all $j \in \mathbb{Z}_{0,2n-1}$. Because of (1.1),

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it follows $[I_{n+1} \otimes (I_p - s_0 s_0^\dagger)] H_n = \text{diag}[0_{np \times nq}, s_{2n} - s_0 s_0^\dagger s_{2n}]$ and $H_n[I_{n+1} \otimes (I_q - s_0^\dagger s_0)] = \text{diag}[0_{np \times nq}, s_{2n} - s_{2n} s_0^\dagger s_0]$. Hence,

$$\begin{aligned} & \left[I_{n+1} \otimes (I_p - s_0 s_0^\dagger) \right] H_n \left[I_{n+1} \otimes (I_q - s_0^\dagger s_0) \right] \\ &= \text{diag}[0_{np \times nq}, s_{2n} - s_{2n} s_0^\dagger s_0 - s_0 s_0^\dagger s_{2n} + s_0 s_0^\dagger s_{2n} s_0^\dagger s_0] \end{aligned}$$

and, taking into account (3.2), (3.1), and $s_0^\# = s_0^\dagger$, we obtain furthermore

$$\left[I_{n+1} \otimes (I_p - s_0 s_0^\dagger) \right] H_n \mathbb{S}_n^\#(I_{n+1} \otimes s_0) = \text{diag}[0_{np \times nq}, s_{2n} s_0^\dagger s_0 - s_0 s_0^\dagger s_{2n} s_0^\dagger s_0]$$

and

$$(I_{n+1} \otimes s_0) \mathbf{S}_n^\# H_n \left[I_{n+1} \otimes (I_q - s_0^\dagger s_0) \right] = \text{diag}[0_{np \times nq}, s_0 s_0^\dagger s_{2n} - s_0 s_0^\dagger s_{2n} s_0^\dagger s_0].$$

Using (7.3), (7.4), and (5.4), we get consequently (9.1). \square

We now turn our attention to block Hankel matrices built from the k -th α -S-transform. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices, and let $k \in \mathbb{Z}_{0,\kappa}$. Then, let

$$H_n^{[k,\alpha]} := [s_{l+m}^{[k,\alpha]}]_{l,m=0}^n \quad (9.2)$$

for all $n \in \mathbb{N}_0$ with $2n \leq \kappa - k$ and let

$$K_n^{[k,\alpha]} := [s_{l+m+1}^{[k,\alpha]}]_{l,m=0}^n \quad (9.3)$$

for all $n \in \mathbb{N}_0$ with $2n + 1 \leq \kappa - k$.

Lemma 9.2. Let $\alpha \in \mathbb{R}$, let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n} \in \mathcal{K}_{q,2n,\alpha}^\geq$. Denote by $(t_j)_{j=0}^{2n-1}$ the first α -S-transform of $(s_j)_{j=0}^{2n}$ and by $(u_j)_{j=0}^{2n-1}$ the $[\alpha]$ -transform of $(t_j)_{j=0}^{2n-1}$. Then

$$\left(\text{diag}[I_q, \mathbf{D}_{n-1}^{(u)}] \right) \mathbf{D}_n H_n \mathbb{D}_n \left(\text{diag}[I_q, \mathbb{D}_{n-1}^{(u)}] \right) = \text{diag}[s_0, H_{n-1}^{[2,\alpha]}] + \Xi_{n,2n} + \Xi_{n,2n-1}^{(t)}.$$

Proof. Because of Proposition 3.8(b), the sequence $(s_j)_{j=0}^{2n}$ belongs to $\tilde{\mathcal{D}}_{q \times q, 2n}$. By virtue of Lemma 9.1, equation (9.1) holds true. In view of (5.4), this implies

$$\mathbf{D}_n H_n \mathbb{D}_n = \text{diag}[s_0, -\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)} + \Xi_{n-1,2n}]. \quad (9.4)$$

Taking into account Theorem 7.21(a) and Proposition 3.8(c), we see that the sequence $(t_j)_{j=0}^{2n-1}$ belongs to $\tilde{\mathcal{D}}_{q \times q, 2n-1}$. Thus, the application of Lemma 7.19(a) (, more precisely, the application of equation (7.9) to the sequence $(t_j)_{j=0}^{2n-1}$), provides us then

$$\mathbf{D}_{n-1}^{(u)} (-\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)} - \Xi_{n-1,2n-1}^{(t)}) \mathbb{D}_{n-1}^{(u)} = H_{n-1}^{(v)},$$

where the sequence $(v_j)_{j=0}^{2n-2}$ is given by $v_j := t_j^{[1,\alpha]}$ for all $j \in \mathbb{Z}_{0,2n-2}$. In view of Remark 7.15 and (5.4), we have

$$\mathbf{D}_{n-1}^{(u)} (\Xi_{n-1,2n-1}^{(t)} + \Xi_{n-1,2n}) \mathbb{D}_{n-1}^{(u)} = \Xi_{n-1,2n-1}^{(t)} + \Xi_{n-1,2n}.$$

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From Definition 8.1 we see that $(v_j)_{j=0}^{2n-2}$ is the second α -S-transform of $(s_j)_{j=0}^{2n}$ which, in view of (9.2), implies $H_{n-1}^{(v)} = H_{n-1}^{[2,\alpha]}$. Hence, we obtain

$$\begin{aligned} & \mathbf{D}_{n-1}^{(u)}(-\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)} + \Xi_{n-1,2n})\mathbb{D}_{n-1}^{(u)} \\ &= \mathbf{D}_{n-1}^{(u)}(-\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)} - \Xi_{n-1,2n-1}^{(t)})\mathbb{D}_{n-1}^{(u)} + \mathbf{D}_{n-1}^{(u)}(\Xi_{n-1,2n-1}^{(t)} + \Xi_{n-1,2n})\mathbb{D}_{n-1}^{(u)} \quad (9.5) \\ &= H_{n-1}^{(v)} + \Xi_{n-1,2n-1}^{(t)} + \Xi_{n-1,2n} = H_{n-1}^{[2,\alpha]} + \Xi_{n-1,2n} + \Xi_{n-1,2n-1}^{(t)} \end{aligned}$$

and, taking (9.4), (9.5), and (5.4) into account, finally

$$\begin{aligned} & \left(\text{diag}[I_q, \mathbf{D}_{n-1}^{(u)}] \right) \mathbf{D}_n H_n \mathbb{D}_n \left(\text{diag}[I_q, \mathbb{D}_{n-1}^{(u)}] \right) \\ &= \text{diag} \left[s_0, \mathbf{D}_{n-1}^{(u)}(-\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)} + \Xi_{n-1,2n})\mathbb{D}_{n-1}^{(u)} \right] \\ &= \text{diag} \left[s_0, H_{n-1}^{[2,\alpha]} + \Xi_{n-1,2n} + \Xi_{n-1,2n-1}^{(t)} \right] = \text{diag} \left[s_0, H_{n-1}^{[2,\alpha]} \right] + \Xi_{n,2n} + \Xi_{n,2n-1}^{(t)}. \quad \square \end{aligned}$$

Lemma 9.3. Let $\alpha \in \mathbb{R}$, let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n+1} \in \mathcal{K}_{q,2n+1,\alpha}^{\geq}$. Denote by $(r_j)_{j=0}^{2n+1}$ the $[+, \alpha]$ -transform of $(s_j)_{j=0}^{2n+1}$ and by $(t_j)_{j=0}^{2n}$ the first α -S-transform of $(s_j)_{j=0}^{2n+1}$. Then

$$\mathbf{D}_n^{(t)} \mathbf{D}_n^{(r)}(-\alpha H_n + K_n)\mathbb{D}_n^{(r)}\mathbb{D}_n^{(t)} = \text{diag}[s_0^{[1,\alpha]}, -\alpha H_{n-1}^{[2,\alpha]} + K_{n-1}^{[2,\alpha]}] + \Xi_{n,2n+1} + \Xi_{n,2n}^{(t)}.$$

Proof. Because of Proposition 3.8(c) the sequence $(s_j)_{j=0}^{2n+1}$ belongs to $\tilde{\mathcal{D}}_{q \times q, 2n+1}$. Thus, by virtue of Lemma 7.19(a), we obtain then the equation $H_n^{[1,\alpha]} = \mathbf{D}_n^{[+, \alpha]}(H_{\alpha \triangleright n} - \Xi_{n,2n+1})\mathbb{D}_n^{[+, \alpha]}$ which, in view of (7.5) and (2.18), implies

$$\mathbf{D}_n^{(r)}(-\alpha H_n + K_n)\mathbb{D}_n^{(r)} = H_n^{[1,\alpha]} + \mathbf{D}_n^{(r)}\Xi_{n,2n+1}\mathbb{D}_n^{(r)}. \quad (9.6)$$

According to Theorem 7.21(a) and Proposition 3.8(b), we have $(t_j)_{j=0}^{2n} \in \tilde{\mathcal{D}}_{q \times q, 2n}$. Using (7.6) and Lemma 9.1, we then obtain

$$\mathbf{D}_n^{(t)} H_n^{[1,\alpha]} \mathbb{D}_n^{(t)} = \mathbf{D}_n^{(t)} H_n^{(t)} \mathbb{D}_n^{(t)} = \text{diag}[t_0, -\alpha H_{n-1}^{(v)} + K_{n-1}^{(v)}] + \Xi_{n,2n}^{(t)}, \quad (9.7)$$

where the sequence $(v_j)_{j=0}^{2n-1}$ is given by $v_j := t_j^{[1,\alpha]}$ for all $j \in \mathbb{Z}_{0,2n-1}$. In view of Remark 7.15 and (5.4), we have

$$\mathbf{D}_n^{(t)} \mathbf{D}_n^{(r)} \Xi_{n,2n+1} \mathbb{D}_n^{(r)} \mathbb{D}_n^{(t)} = \mathbf{D}_n^{(t)} \Xi_{n,2n+1} \mathbb{D}_n^{(t)} = \Xi_{n,2n+1}. \quad (9.8)$$

From Definition 8.1 we see that $(v_j)_{j=0}^{2n-1}$ is the second α -S-transform of $(s_j)_{j=0}^{2n+1}$ which, in view of (9.2) and (9.3), implies

$$-\alpha H_{n-1}^{(v)} + K_{n-1}^{(v)} = -\alpha H_{n-1}^{[2,\alpha]} + K_{n-1}^{[2,\alpha]}. \quad (9.9)$$

Finally, using (9.6), (9.7), (9.8), and (9.9) we obtain then

$$\begin{aligned} \mathbf{D}_n^{(t)} \mathbf{D}_n^{(r)}(-\alpha H_n + K_n)\mathbb{D}_n^{(r)}\mathbb{D}_n^{(t)} &= \mathbf{D}_n^{(t)} H_n^{[1,\alpha]} \mathbb{D}_n^{(t)} + \mathbf{D}_n^{(t)} \mathbf{D}_n^{(r)} \Xi_{n,2n+1} \mathbb{D}_n^{(r)} \mathbb{D}_n^{(t)} \\ &= \text{diag}[t_0, -\alpha H_{n-1}^{(v)} + K_{n-1}^{(v)}] + \Xi_{n,2n}^{(t)} + \Xi_{n,2n+1} \\ &= \text{diag}[s_0^{[1,\alpha]}, -\alpha H_{n-1}^{[2,\alpha]} + K_{n-1}^{[2,\alpha]}] + \Xi_{n,2n+1} + \Xi_{n,2n}^{(t)}. \quad \square \end{aligned}$$

9. On the right α -Stieltjes parametrization against to the background of the Schur-type algorithm

Now we introduce further particular matrices, which play an essential role in our further considerations. We will see that these matrices indicate in some sense how far a sequence $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^\geq$ is away from the class $\mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. Then let

$$\epsilon_{j,k} := s_j^{[k,\alpha]} - s_0^{[k,\alpha]}(s_0^{[k,\alpha]})^\dagger s_j^{[k,\alpha]}(s_0^{[k,\alpha]})^\dagger s_0^{[k,\alpha]} \quad (9.10)$$

for all $k \in \mathbb{Z}_{0,\kappa}$ and all $j \in \mathbb{Z}_{0,\kappa-k}$. Furthermore, for all $m \in \mathbb{Z}_{0,\kappa}$ and all $l \in \mathbb{Z}_{m,\kappa}$, let

$$\rho_{l,m} := \begin{cases} 0_{q \times q} & \text{if } m = 0 \\ \sum_{k=0}^{m-1} \epsilon_{l-k,k} & \text{if } m \geq 1 \end{cases}. \quad (9.11)$$

Lemma 9.4. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. Then $\epsilon_{j,k} = 0_{q \times q}$ for all $k \in \mathbb{Z}_{0,\kappa}$ and all $j \in \mathbb{Z}_{0,\kappa-k}$. Furthermore, $\rho_{l,m} = 0_{q \times q}$ for all $m \in \mathbb{Z}_{0,\kappa}$ and all $l \in \mathbb{Z}_{m,\kappa}$.*

Proof. Let $k \in \mathbb{Z}_{0,\kappa}$. According to Theorem 8.10(b), we have $(s_j^{[k,\alpha]})_{j=0}^{\kappa-k} \in \mathcal{K}_{q,\kappa-k,\alpha}^{\geq,e}$, which, in view of Proposition 3.8(a), implies $(s_j^{[k,\alpha]})_{j=0}^{\kappa-k} \in \mathcal{D}_{q \times q, \kappa-k}$. From (9.10), Definition 3.3, and parts (c) and (b) of Remark A.1, we obtain then $\epsilon_{j,k} = 0_{q \times q}$ for all $j \in \mathbb{Z}_{0,\kappa-k}$. In view of (9.11), this implies $\rho_{l,m} = 0_{q \times q}$ for all $m \in \mathbb{Z}_{0,\kappa}$ and all $l \in \mathbb{Z}_{m,\kappa}$. \square

Remark 9.5. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^\geq$. Because of (9.10), (9.11), Remark 8.2, and Lemma 9.4, then $\epsilon_{j,k} = 0_{q \times q}$ for all $k \in \mathbb{Z}_{0,\kappa-1}$ and all $j \in \mathbb{Z}_{0,\kappa-1-k}$ and, furthermore, $\rho_{l,m} = 0_{q \times q}$ for all $m \in \mathbb{Z}_{0,\kappa-1}$ and all $l \in \mathbb{Z}_{m,\kappa-1}$.

Lemma 9.6. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices. For all $k \in \mathbb{Z}_{0,\kappa-1}$, then $-\alpha s_0^{[k,\alpha]} + s_1^{[k,\alpha]} = s_0^{[k+1,\alpha]} + \epsilon_{1,k}$.*

Proof. Let $k \in \mathbb{Z}_{0,\kappa-1}$. Denote by $(t_j)_{j=0}^{\kappa-k}$ the k -th α -S-transform of $(s_j)_{j=0}^\kappa$. The application of Lemma 7.8 to the sequence $(t_j)_{j=0}^{\kappa-k}$ yields then $t_0^{[1,\alpha]} = t_0 t_0^\dagger t_1^{[+, \alpha]} t_0^\dagger t_0$ which, in view of Definition 4.1, implies

$$t_0^{[1,\alpha]} = t_0 t_0^\dagger (-\alpha t_0 + t_1) t_0^\dagger t_0 = -\alpha t_0 t_0^\dagger t_0 t_0^\dagger t_0 + t_0 t_0^\dagger t_1 t_0^\dagger t_0 = -\alpha t_0 + t_0 t_0^\dagger t_1 t_0^\dagger t_0.$$

Taking additionally Definition 8.1 and (9.10) into account, this implies

$$-\alpha s_0^{[k,\alpha]} + s_1^{[k,\alpha]} = -\alpha t_0 + t_1 = t_0^{[1,\alpha]} - t_0 t_0^\dagger t_1 t_0^\dagger t_0 + t_1 = s_0^{[k+1,\alpha]} + \epsilon_{1,k}. \quad \square$$

Remark 9.7. Let $\alpha \in \mathbb{C}$ and let $(s_j)_{j=0}^0$ be a sequence of complex $p \times q$ matrices. In view of (1.1), Definition 8.1, and (9.11), then $H_0 = s_0^{[0,\alpha]} + \rho_{0,0}$.

Remark 9.8. Let $\alpha \in \mathbb{C}$ and let $(s_j)_{j=0}^1$ be a sequence from $\mathbb{C}^{p \times q}$. In view of (1.1), (1.2), Definition 8.1, Lemma 9.6, and (9.11), then $-\alpha H_0 + K_0 = s_0^{[1,\alpha]} + \rho_{1,1}$.

The following two lemmas play a key role in the proof of central results of this paper, because they provide the block LDU decompositions we are striving for. Note that the sets $\mathfrak{L}_{q,n}$ and $\mathfrak{U}_{q,n}$ were introduced in Notation 7.14.

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Lemma 9.9. Let $\alpha \in \mathbb{R}$, let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n} \in \mathcal{K}_{q,2n,\alpha}^{\geq}$. For all $k \in \mathbb{Z}_{0,2n}$, denote by $(t_j^{(k)})_{j=0}^{2n-k}$ the k -th α -S-transform of $(s_j)_{j=0}^{2n}$ and by $(u_j^{(k)})_{j=0}^{2n-k}$ the $[\alpha, +]$ -transform of $(t_j^{(k)})_{j=0}^{2n-k}$. For all $l \in \mathbb{Z}_{0,n-1}$, let

$$\mathbf{V}_l := \begin{cases} \text{diag}[I_q, \mathbf{D}_{n-1}^{(u^{(2l+1)})}] \cdot \mathbf{D}_n^{(t^{(2l)})} & \text{if } l = 0 \\ \text{diag}[I_q, \text{diag}[I_q, \mathbf{D}_{n-l-1}^{(u^{(2l+1)})}] \cdot \mathbf{D}_{n-l}^{(t^{(2l)})}] & \text{if } l \geq 1 \end{cases}$$

and

$$\mathbb{V}_l := \begin{cases} \mathbb{D}_n^{(t^{(2l)})} \cdot \text{diag}[I_q, \mathbb{D}_{n-1}^{(u^{(2l+1)})}] & \text{if } l = 0 \\ \text{diag}[I_q, \mathbb{D}_{n-l}^{(t^{(2l)})} \cdot \text{diag}[I_q, \mathbb{D}_{n-l-1}^{(u^{(2l+1)})}]] & \text{if } l \geq 1 \end{cases}.$$

(a) For all $m \in \mathbb{Z}_{1,n}$, the matrix $\tilde{\mathbf{V}}_m := \mathbf{V}_{m-1} \mathbf{V}_{m-2} \cdots \mathbf{V}_0$ belongs to $\mathfrak{L}_{q,n}$ and the matrix $\tilde{\mathbb{V}}_m := \mathbb{V}_0 \mathbb{V}_1 \cdots \mathbb{V}_{m-1}$ belongs to $\mathfrak{U}_{q,n}$.

(b) $\tilde{\mathbf{V}}_n H_n \tilde{\mathbf{V}}_n = \text{diag}[s_0^{[0,\alpha]}, s_0^{[2,\alpha]}, \dots, s_0^{[2(n-1),\alpha]}, s_0^{[2n,\alpha]} + \rho_{2n,2n}]$.

(c) If $(s_j)_{j=0}^{2n} \in \mathcal{K}_{q,2n,\alpha}^{\geq,e}$, then $\tilde{\mathbf{V}}_n H_n \tilde{\mathbf{V}}_n = \text{diag}[s_0^{[0,\alpha]}, s_0^{[2,\alpha]}, \dots, s_0^{[2n,\alpha]}]$.

Proof. (a) Using Remarks 7.15, A.2, and A.3, we easily see that $\mathbf{V}_l \in \mathfrak{L}_{q,n}$ and that $\mathbb{V}_l \in \mathfrak{U}_{q,n}$ hold true for all $l \in \mathbb{Z}_{0,n-1}$ which, in view of Remark A.2, implies $\tilde{\mathbf{V}}_n \in \mathfrak{L}_{q,n}$ and $\tilde{\mathbb{V}}_n \in \mathfrak{U}_{q,n}$.

(b) We prove (b) by induction. In view of (5.4), (9.10), and (9.11), there is, according to Lemma 9.2 and Definition 8.1, some $m \in \mathbb{Z}_{1,n}$ such that for all $k \in \mathbb{Z}_{1,m}$ the following equation holds:

$$(I_k) \quad \tilde{\mathbf{V}}_k H_n \tilde{\mathbf{V}}_k = \text{diag}[s_0^{[0,\alpha]}, s_0^{[2,\alpha]}, \dots, s_0^{[2(k-1),\alpha]}, H_{n-k}^{[2k,\alpha]}] + \text{diag}[0_{nq \times nq}, \rho_{2n,2k}].$$

According to (9.2), we have furthermore

$$H_0^{[2n,\alpha]} = s_0^{[2n,\alpha]}. \quad (9.12)$$

If $m = n$, then (b) immediately follows from (I_m) and (9.12).

Now we consider the case $m < n$. Obviously, using (I_m) , we easily see that

$$\begin{aligned} \tilde{\mathbf{V}}_{m+1} H_n \tilde{\mathbf{V}}_{m+1} &= \mathbf{V}_m \tilde{\mathbf{V}}_m H_n \tilde{\mathbf{V}}_m \mathbb{V}_m \\ &= \text{diag}[s_0^{[0,\alpha]}, s_0^{[2,\alpha]}, \dots, s_0^{[2(m-1),\alpha]}] \\ &\quad \text{diag}[I_q, \mathbf{D}_{n-m-1}^{(u^{(2m+1)})}] \mathbf{D}_{n-m}^{(t^{(2m)})} H_{n-m}^{[2m,\alpha]} \mathbb{D}_{n-m}^{(t^{(2m)})} \text{diag}[I_q, \mathbb{D}_{n-m-1}^{(u^{(2m+1)})}] \\ &\quad + \mathbf{V}_m \text{diag}[0_{nq \times nq}, \rho_{2n,2m}] \mathbb{V}_m. \end{aligned} \quad (9.13)$$

Due to Theorem 8.10(a), the sequence $(s_j^{[2m,\alpha]})_{j=0}^{2(n-m)}$ belongs to $\mathcal{K}_{q,2(n-m),\alpha}^{\geq}$. The application of Lemma 9.2 to the sequence $(s_j^{[2m,\alpha]})_{j=0}^{2(n-m)}$ yields, in view of (9.2) and Definition 8.1, then

$$\begin{aligned} &\text{diag}[I_q, \mathbf{D}_{n-m-1}^{(u^{(2m+1)})}] \mathbf{D}_{n-m}^{(t^{(2m)})} H_{n-m}^{[2m,\alpha]} \mathbb{D}_{n-m}^{(t^{(2m)})} \text{diag}[I_q, \mathbb{D}_{n-m-1}^{(u^{(2m+1)})}] \\ &= \text{diag}[s_0^{[2m,\alpha]}, H_{n-m-1}^{[2m+2,\alpha]}] + \Xi_{n-m,2(n-m)}^{(t^{(2m)})} + \Xi_{n-m,2(n-m)-1}^{(t^{(2m+1)})}. \end{aligned} \quad (9.14)$$

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Because of $\mathbf{V}_m \in \mathfrak{L}_{q,n}$ and $\mathbb{V}_m \in \mathfrak{U}_{q,n}$, we have furthermore

$$\mathbf{V}_m \text{diag}[0_{nq \times nq}, \rho_{2n,2m}] \mathbb{V}_m = \text{diag}[0_{nq \times nq}, \rho_{2n,2m}]. \quad (9.15)$$

Taking into account (9.13), (9.14), (9.15), and (5.4), thus

$$\begin{aligned} \tilde{\mathbf{V}}_{m+1} H_n \tilde{\mathbb{V}}_{m+1} &= \text{diag}[s_0^{[0,\alpha]}, s_0^{[2,\alpha]}, \dots, s_0^{[2(m-1),\alpha]}, s_0^{[2m,\alpha]}, H_{n-(m+1)}^{[2(m+1),\alpha]}] \\ &\quad + \Xi_{n,2(n-m)}^{(t^{(2m)})} + \Xi_{n,2(n-m)-1}^{(t^{(2m+1)})} + \text{diag}[0_{nq \times nq}, \rho_{2n,2m}]. \end{aligned} \quad (9.16)$$

In view of (5.4), (9.10), and (9.11), we have moreover

$$\Xi_{n,2(n-m)}^{(t^{(2m)})} + \Xi_{n,2(n-m)-1}^{(t^{(2m+1)})} + \text{diag}[0_{nq \times nq}, \rho_{2n,2m}] = \text{diag}[0_{nq \times nq}, \rho_{2n,2(m+1)}]. \quad (9.17)$$

From (9.16) and (9.17) it follows (\mathbf{I}_{m+1}) . Hence, (\mathbf{I}_n) is inductively proved. In view of (9.12), the proof is of (b) finished.

(c) Combine (b) and Lemma 9.4. \square

Lemma 9.10. Let $\alpha \in \mathbb{R}$, let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n+1} \in \mathcal{K}_{q,2n+1,\alpha}^{\geq}$. For all $k \in \mathbb{Z}_{0,2n+1}$, denote by $(t_j^{(k)})_{j=0}^{2n+1-k}$ the k -th α -S-transform of $(s_j)_{j=0}^{2n+1}$ and by $(u_j^{(k)})_{j=0}^{2n+1-k}$ the $[\alpha, \alpha]$ -transform of $(t_j^{(k)})_{j=0}^{2n+1-k}$. For all $l \in \mathbb{Z}_{0,n-1}$, let

$$\mathbf{W}_l := \begin{cases} \mathbf{D}_n^{(t^{(2l+1)})} \mathbf{D}_n^{(u^{(2l)})} & \text{if } l = 0 \\ \text{diag}[I_{lq}, \mathbf{D}_{n-l}^{(t^{(2l+1)})} \mathbf{D}_{n-l}^{(u^{(2l)})}] & \text{if } l \geq 1 \end{cases}$$

and

$$\mathbb{W}_l := \begin{cases} \mathbb{D}_n^{(u^{(2l)})} \mathbb{D}_n^{(t^{(2l+1)})} & \text{if } l = 0 \\ \text{diag}[I_{lq}, \mathbb{D}_{n-l}^{(u^{(2l)})} \mathbb{D}_{n-l}^{(t^{(2l+1)})}] & \text{if } l \geq 1 \end{cases}.$$

(a) For all $m \in \mathbb{Z}_{1,n}$, the matrices $\tilde{\mathbf{W}}_m := \mathbf{W}_{m-1} \mathbf{W}_{m-2} \cdots \mathbf{W}_0$ and $\tilde{\mathbb{W}}_m := \mathbb{W}_0 \mathbb{W}_1 \cdots \mathbb{W}_{m-1}$ belong to $\mathfrak{L}_{q,n}$ and to $\mathfrak{U}_{q,n}$, respectively.

(b) $\tilde{\mathbf{W}}_n(-\alpha H_n + K_n) \tilde{\mathbb{W}}_n = \text{diag}[s_0^{[1,\alpha]}, s_0^{[3,\alpha]}, \dots, s_0^{[2n-1,\alpha]}, s_0^{[2n+1,\alpha]} + \rho_{2n+1,2n+1}].$

(c) If $(s_j)_{j=0}^{2n+1} \in \mathcal{K}_{q,2n+1,\alpha}^{\geq, e}$, then $\tilde{\mathbf{W}}_n(-\alpha H_n + K_n) \tilde{\mathbb{W}}_n = \text{diag}[s_0^{[1,\alpha]}, s_0^{[3,\alpha]}, \dots, s_0^{[2n+1,\alpha]}].$

Proof. (a) Using Remarks 7.15, A.2, and A.3, we easily see that $\mathbf{W}_l \in \mathfrak{L}_{q,n}$ and that $\mathbb{W}_l \in \mathfrak{U}_{q,n}$ hold true for all $l \in \mathbb{Z}_{0,n-1}$ which, in view of Remark A.2, implies $\tilde{\mathbf{W}}_n \in \mathfrak{L}_{q,n}$ and $\tilde{\mathbb{W}}_n \in \mathfrak{U}_{q,n}$.

(b) We prove (b) by induction. According to Lemma 9.3, there is, in view of (5.4), (9.10), and (9.11), some $m \in \mathbb{Z}_{1,n}$ such that for all $k \in \mathbb{Z}_{1,m}$ the following equation holds:

$$(\mathbf{I}_k) \quad \tilde{\mathbf{W}}_k H_n \tilde{\mathbb{W}}_k = \text{diag}[s_0^{[1,\alpha]}, \dots, s_0^{[2k-1,\alpha]}, -\alpha H_{n-k}^{[2k,\alpha]} + K_{n-k}^{[2k,\alpha]}] + \text{diag}[0_{nq \times nq}, \rho_{2n+1,2k}].$$

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The application of Lemma 9.6 yields $-\alpha s_0^{[2n,\alpha]} + s_1^{[2n,\alpha]} = s_0^{[2n+1,\alpha]} + \epsilon_{1,2n}$. Taking additionally into account (9.2) and (9.3), we obtain then

$$-\alpha H_0^{[2n,\alpha]} + K_0^{[2n,\alpha]} = -\alpha s_0^{[2n,\alpha]} + s_1^{[2n,\alpha]} = s_0^{[2n+1,\alpha]} + \epsilon_{1,2n}. \quad (9.18)$$

In view of (9.11), we have furthermore

$$\epsilon_{1,2n} + \rho_{2n+1,2n} = \rho_{2n+1,2n+1}. \quad (9.19)$$

If $m = n$, then (b) immediately follows from (I_m) , (9.18), and (9.19).

Now we consider the case $m < n$. Obviously, from (I_m) we have

$$\begin{aligned} \tilde{\mathbf{W}}_{m+1} H_n \tilde{\mathbb{W}}_{m+1} &= \mathbf{W}_m \tilde{\mathbf{W}}_m H_n \tilde{\mathbb{W}}_m \mathbb{W}_m \\ &= \text{diag} \left[s_0^{[1,\alpha]}, s_0^{[3,\alpha]}, \dots, s_0^{[2m-1,\alpha]}, \right. \\ &\quad \left. \mathbf{D}_{n-m}^{(t^{(2m+1)})} \mathbf{D}_{n-m}^{(u^{(2m)})} (-\alpha H_{n-m}^{[2m,\alpha]} + K_{n-m}^{[2m,\alpha]}) \mathbb{D}_{n-m}^{(u^{(2m)})} \mathbb{D}_{n-m}^{(t^{(2m+1)})} \right] \\ &\quad + \mathbf{W}_m \text{diag}[0_{nq \times nq}, \rho_{2n+1,2m}] \mathbb{W}_m. \end{aligned} \quad (9.20)$$

According to Theorem 8.10(a), the sequence $(s_j^{[2m,\alpha]})_{j=0}^{2(n-m)+1}$ belongs to $\mathcal{K}_{q,2(n-m)+1,\alpha}^{\geq}$. The application of Lemma 9.3 to the sequence $(s_j^{[2m,\alpha]})_{j=0}^{2(n-m)+1}$ yields, in view of (9.2), (9.3), and Definition 8.1, then

$$\begin{aligned} &\mathbf{D}_{n-m}^{(t^{(2m+1)})} \mathbf{D}_{n-m}^{(u^{(2m)})} (-\alpha H_{n-m}^{[2m,\alpha]} + K_{n-m}^{[2m,\alpha]}) \mathbb{D}_{n-m}^{(u^{(2m)})} \mathbb{D}_{n-m}^{(t^{(2m+1)})} \\ &= \text{diag}[s_0^{[2m+1,\alpha]}, -\alpha H_{n-m-1}^{[2m+2,\alpha]} + K_{n-m-1}^{[2m+2,\alpha]}] + \Xi_{n-m,2(n-m)+1}^{(t^{(2m)})} + \Xi_{n-m,2(n-m)}^{(t^{(2m+1)})}. \end{aligned}$$

Because of $\mathbf{W}_m \in \mathfrak{L}_{q,n}$ and $\mathbb{W}_m \in \mathfrak{U}_{q,n}$, furthermore we get $\mathbf{W}_m \cdot \text{diag}[0_{nq \times nq}, \rho_{2n+1,2m}] \mathbb{W}_m = \text{diag}[0_{nq \times nq}, \rho_{2n+1,2m}]$. Taking into account (9.20) and (5.4), thus we have

$$\begin{aligned} \tilde{\mathbf{W}}_{m+1} H_n \tilde{\mathbb{W}}_{m+1} &= \text{diag}[s_0^{[1,\alpha]}, s_0^{[3,\alpha]}, \dots, s_0^{[2m-1,\alpha]}, s_0^{[2m+1,\alpha]}, -\alpha H_{n-(m+1)}^{[2(m+1),\alpha]} + K_{n-(m+1)}^{[2(m+1),\alpha]}] \\ &\quad + \Xi_{n,2(n-m)+1}^{(t^{(2m)})} + \Xi_{n,2(n-m)}^{(t^{(2m+1)})} + \text{diag}[0_{nq \times nq}, \rho_{2n+1,2m}]. \end{aligned} \quad (9.21)$$

In view of (5.4), (9.10), and (9.11), we have moreover

$$\Xi_{n,2(n-m)+1}^{(t^{(2m)})} + \Xi_{n,2(n-m)}^{(t^{(2m+1)})} + \text{diag}[0_{nq \times nq}, \rho_{2n+1,2m}] = \text{diag}[0_{nq \times nq}, \rho_{2n+1,2(m+1)}]. \quad (9.22)$$

From (9.21) and (9.22) it follows (I_{m+1}) . Hence, (I_n) is inductively proved. In view of (9.18) and (9.19), the proof of (b) is finished.

(c) Combine (b) and Lemma 9.4. \square

Remark 9.11. Let $n \in \mathbb{N}$ and let $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$. Then, according to [20, Proposition 4.17], there exists a matrix $\mathbb{F}_n \in \mathfrak{U}_{q,n}$ such that

$$\mathbb{F}_n^* H_n \mathbb{F}_n = \text{diag}[L_0, L_1, \dots, L_n]. \quad (9.23)$$

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The following Theorems 9.12 and 9.15 indicate an explicit connection between the α -S-transforms and the right α -Stieltjes parametrization. These results can be considered as the first two main results of this paper.

Theorem 9.12. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^>$. Then $Q_m = s_0^{[m,\alpha]} + \rho_{m,m}$ and, in the case $m \geq 1$, furthermore $Q_j = s_0^{[j,\alpha]}$ for all $j \in \mathbb{Z}_{0,m-1}$.*

Proof. First observe that, in view of Definition 2.8, (2.11), (1.1), and Remark 9.7, we have

$$Q_0 = L_0 = s_0 = H_0 = s_0^{[0,\alpha]} + \rho_{0,0}. \quad (9.24)$$

In the case $m = 0$, the proof is complete. In the case $m \geq 1$, from Definition 2.8, (2.16), (2.11), (1.3), (1.1), (1.2), and Remark 9.8, we see

$$Q_1 = L_{\alpha \triangleright 0} = s_{\alpha \triangleright 0} = -\alpha s_0 + s_1 = -\alpha H_0 + K_0 = s_0^{[1,\alpha]} + \rho_{1,1} \quad (9.25)$$

and, in view of Remark 9.5, furthermore

$$\rho_{m-1,m-1} = 0_{q \times q}. \quad (9.26)$$

In the case $m = 1$, the assertion follows from (9.25), (9.24), and (9.26).

Now we consider the case $m = 2n$ with some $n \in \mathbb{N}$. Then, from (1.4) we obtain $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^>$. Thus, in view of Remark 9.11, there exists a matrix $\mathbb{F}_n \in \mathfrak{U}_{q,n}$ such that (9.23) holds true. Taking into account Definition 2.8, we get then

$$\mathbb{F}_n^* H_n \mathbb{F}_n = \text{diag}[Q_0, Q_2, \dots, Q_{2n}]. \quad (9.27)$$

Furthermore, according to Lemma 9.9, there exist matrices $\tilde{\mathbf{V}}_n \in \mathfrak{L}_{q,n}$ and $\tilde{\mathbf{V}}_n \in \mathfrak{U}_{q,n}$ such that

$$\tilde{\mathbf{V}}_n H_n \tilde{\mathbf{V}}_n = \text{diag}[s_0^{[0,\alpha]}, s_0^{[2,\alpha]}, \dots, s_0^{[2(n-1),\alpha]}, s_0^{[2n,\alpha]} + \rho_{2n,2n}]. \quad (9.28)$$

Since because of Remark A.2 the matrices \mathbb{F}_n^* , $\tilde{\mathbf{V}}_n$, \mathbb{F}_n , and $\tilde{\mathbf{V}}_n$ are non-singular with $\{\mathbb{F}_n^*, \tilde{\mathbf{V}}_n^{-1}\} \subseteq \mathfrak{L}_{q,n}$ and $\{\mathbb{F}_n^{-1}, \tilde{\mathbf{V}}_n\} \subseteq \mathfrak{U}_{q,n}$, we obtain from (9.27), (9.28), and Remark A.4 that $Q_{2k} = s_0^{[2k,\alpha]}$ holds true for all $k \in \mathbb{Z}_{0,n-1}$ and that $Q_{2n} = s_0^{[2n,\alpha]} + \rho_{2n,2n}$. In the case $n = 1$, the assertion hence follows from (9.25) and (9.26). Now we consider the case $n \geq 2$. From (1.4) we obtain $(s_{\alpha \triangleright j})_{j=0}^{2(n-1)} \in \mathcal{H}_{q,2(n-1)}^>$. Thus, in view of Remark 9.11, (2.15), and (2.16), there exists a matrix $\mathbb{G}_{n-1} \in \mathfrak{U}_{q,n-1}$ such that $\mathbb{G}_{n-1}^* H_{\alpha \triangleright n-1} \mathbb{G}_{n-1} = \text{diag}[L_{\alpha \triangleright 0}, L_{\alpha \triangleright 1}, \dots, L_{\alpha \triangleright n-1}]$. From (2.18) and Definition 2.8, we get then

$$\mathbb{G}_{n-1}^* (-\alpha H_{n-1} + K_n) \mathbb{G}_{n-1} = \text{diag}[Q_1, Q_3, \dots, Q_{2n-1}]. \quad (9.29)$$

In view of Remark 2.1, we have $(s_j)_{j=0}^{2n-1} \in \mathcal{K}_{q,2n-1,\alpha}^>$. Thus, according to Lemma 9.10, there exist matrices $\tilde{\mathbf{W}}_{n-1} \in \mathfrak{L}_{q,n-1}$ and $\tilde{\mathbf{W}}_{n-1} \in \mathfrak{U}_{q,n-1}$ such that

$$\begin{aligned} \tilde{\mathbf{W}}_{n-1} (-\alpha H_{n-1} + K_n) \tilde{\mathbf{W}}_{n-1} \\ = \text{diag}[s_0^{[1,\alpha]}, s_0^{[3,\alpha]}, \dots, s_0^{[2n-3,\alpha]}, s_0^{[2n-1,\alpha]} + \rho_{2n-1,2n-1}]. \end{aligned} \quad (9.30)$$

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Since because of Remark A.2 the matrices \mathbb{G}_{n-1}^* , $\tilde{\mathbf{W}}_{n-1}$, \mathbb{G}_{n-1} , and $\tilde{\mathbb{W}}_{n-1}$ are non-singular with $\{\mathbb{G}_{n-1}^{-*}, \tilde{\mathbf{W}}_{n-1}^{-1}\} \subseteq \mathfrak{L}_{q,n-1}$ and $\{\mathbb{G}_{n-1}^{-1}, \tilde{\mathbb{W}}_{n-1}^{-1}\} \subseteq \mathfrak{U}_{q,n-1}$, we obtain from (9.29), (9.30), and Remark A.4 that $Q_{2k+1} = s_0^{[2k+1,\alpha]}$ holds true for all $k \in \mathbb{Z}_{0,n-2}$ and that $Q_{2n-1} = s_0^{[2n-1,\alpha]} + \rho_{2n-1,2n-1}$ which, in view of (9.26), completes the proof in this case.

Finally, we consider the case where $m = 2n + 1$ with some $n \in \mathbb{N}$. Then, from (1.5) we obtain $(s_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$, whereas Remark 2.1 yields $(s_j)_{j=0}^{2n} \in \mathcal{K}_{q,2n,\alpha}^{\geq}$. Similar to the above mentioned case, we conclude that $Q_{2k} = s_0^{[2k,\alpha]}$ holds true for all $k \in \mathbb{Z}_{0,n-1}$ and that $Q_{2n} = s_0^{[2n,\alpha]} + \rho_{2n,2n}$ which, in view of (9.26), implies $Q_{2k} = s_0^{[2k,\alpha]}$ for all $k \in \mathbb{Z}_{0,n}$. Moreover, from (1.5) we obtain $(s_{\alpha \triangleright j})_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$. Again similar to the above mentioned case, we conclude that $Q_{2k+1} = s_0^{[2k+1,\alpha]}$ holds true for all $k \in \mathbb{Z}_{0,n-1}$ and that $Q_{2n+1} = s_0^{[2n+1,\alpha]} + \rho_{2n+1,2n+1}$. This completes the proof. \square

Corollary 9.13. *Let $\alpha \in \mathbb{R}$, let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n} \in \mathcal{K}_{q,2n,\alpha}^{\geq}$. For all $k \in \mathbb{Z}_{0,2n}$ denote by $(s_j^{[k,\alpha]})_{j=0}^{2n-k}$ the k -th α -S-transform of $(s_j)_{j=0}^{2n}$. Then*

$$\text{rank } H_n = \left(\sum_{k=0}^{n-1} \text{rank } s_0^{[2k,\alpha]} \right) + \text{rank}(s_0^{[2n,\alpha]} + \rho_{2n,2n})$$

and

$$\det H_n = \left(\prod_{k=0}^{n-1} \det s_0^{[2k,\alpha]} \right) \det(s_0^{[2n,\alpha]} + \rho_{2n,2n}).$$

Furthermore,

$$\text{rank } H_{\alpha \triangleright n-1} = \sum_{k=0}^{n-1} \text{rank } s_0^{[2k+1,\alpha]} \quad \text{and} \quad \det H_{\alpha \triangleright n-1} = \prod_{k=0}^{n-1} \det s_0^{[2k+1,\alpha]}.$$

Proof. From [21, Lemma 4.11(a)] we get the equations $\text{rank } H_n = \sum_{k=0}^n \text{rank } Q_{2k}$ and $\det H_n = \prod_{k=0}^n \det Q_{2k}$. In view of [21, Lemma 4.11(b)], we see furthermore that $\text{rank } H_{\alpha \triangleright n-1} = \sum_{k=0}^{n-1} \text{rank } Q_{2k+1}$ and $\det H_{\alpha \triangleright n-1} = \prod_{k=0}^{n-1} \det Q_{2k+1}$. Applying Theorem 9.12, we obtain the asserted equations. \square

Corollary 9.14. *Let $\alpha \in \mathbb{R}$, let $n \in \mathbb{N}$, and let $(s_j)_{j=0}^{2n+1} \in \mathcal{K}_{q,2n+1,\alpha}^{\geq}$. For all $k \in \mathbb{Z}_{0,2n+1}$ denote by $(s_j^{[k,\alpha]})_{j=0}^{2n+1-k}$ the k -th α -S-transform of $(s_j)_{j=0}^{2n+1}$. Then*

$$\text{rank } H_n = \sum_{k=0}^n \text{rank } s_0^{[2k,\alpha]} \quad \text{and} \quad \det H_n = \prod_{k=0}^n \det s_0^{[2k,\alpha]}.$$

Furthermore,

$$\text{rank } H_{\alpha \triangleright n} = \left(\sum_{k=0}^{n-1} \text{rank } s_0^{[2k+1,\alpha]} \right) + \text{rank}(s_0^{[2n+1,\alpha]} + \rho_{2n+1,2n+1})$$

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and

$$\det H_{\alpha \triangleright n} = \left(\prod_{k=0}^{n-1} \det s_0^{[2k+1, \alpha]} \right) \det(s_0^{[2n+1, \alpha]} + \rho_{2n+1, 2n+1}).$$

Proof. From [21, Lemma 4.11(a)] we get the equations $\text{rank } H_n = \sum_{k=0}^n \text{rank } Q_{2k}$ and $\det H_n = \prod_{k=0}^n \det Q_{2k}$. In view of [21, Lemma 4.11(b)], we see furthermore that $\text{rank } H_{\alpha \triangleright n} = \sum_{k=0}^n \text{rank } Q_{2k+1}$ and $\det H_{\alpha \triangleright n} = \prod_{k=0}^n \det Q_{2k+1}$. Using Theorem 9.12, we obtain the asserted equations. \square

Now we obtain the second main result of this paper, which indicates that Theorem 9.12 can be simplified for the subclass $\mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$:

Theorem 9.15. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$. Then $(s_0^{[j, \alpha]})_{j=0}^\kappa$ is exactly the right α -Stieltjes parametrization of $(s_j)_{j=0}^\kappa$.*

Proof. Obviously, we have $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$. Denote by $(Q_j)_{j=0}^\kappa$ the right α -Stieltjes parametrization of $(s_j)_{j=0}^\kappa$.

- (i) First we consider the case $\kappa \in \mathbb{N}_0$. Because of $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$ and Theorem 9.12, then $Q_\kappa = s_0^{[\kappa, \alpha]} + \rho_{\kappa, \kappa}$ and, in the case $\kappa \geq 1$, furthermore $Q_j = s_0^{[j, \alpha]}$ for all $j \in \mathbb{Z}_{0, \kappa-1}$. In view of $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$ and Lemma 9.4, we obtain moreover $\rho_{\kappa, \kappa} = 0_{q \times q}$, which completes the proof in this case.
- (ii) Finally, we consider the case $\kappa = +\infty$. Let $l \in \mathbb{N}_0$ and let the sequence $(r_j)_{j=0}^l$ be given by $r_j := s_j$ for all $j \in \mathbb{Z}_{0, l}$. In view of $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$, then $(r_j)_{j=0}^l \in \mathcal{K}_{q, l, \alpha}^{\geq, e}$. For all $k \in \mathbb{Z}_{0, l}$, denote by $(v_j^{(k)})_{j=0}^{l-k}$ the k -th α -S-transform of $(r_j)_{j=0}^l$. By virtue of Remark 8.2, then $v_0^{(l)} = s_0^{[l, \alpha]}$. According to the above already proved (i), we get, in view of $l \in \mathbb{N}_0$ and $(r_j)_{j=0}^l \in \mathcal{K}_{q, l, \alpha}^{\geq, e}$, furthermore $R_l = v_0^{(l)}$, where $(R_j)_{j=0}^l$ denotes the right α -Stieltjes parametrization of $(r_j)_{j=0}^l$. By Remark 2.10, moreover $R_l = Q_l$. Hence, $s_0^{[l, \alpha]} = v_0^{(l)} = R_l = Q_l$ for all $l \in \mathbb{N}_0$. The proof is complete. \square

Corollary 9.16. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$. For all $k \in \mathbb{Z}_{0, \kappa}$ denote by $(s_j^{[k, \alpha]})_{j=0}^{\kappa-k}$ the k -th α -S-transform of $(s_j)_{j=0}^\kappa$. Then $\text{rank } H_n = \sum_{k=0}^n \text{rank } s_0^{[2k, \alpha]}$ and $\det H_n = \prod_{k=0}^n \det s_0^{[2k, \alpha]}$ for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$ and $\text{rank } H_{\alpha \triangleright n} = \sum_{k=0}^n \text{rank } s_0^{[2k+1, \alpha]}$ and $\det H_{\alpha \triangleright n} = \prod_{k=0}^n \det s_0^{[2k+1, \alpha]}$ for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$.*

Proof. Obviously, we have $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$. From [21, Lemma 4.11] we get then that $\text{rank } H_n = \sum_{k=0}^n \text{rank } Q_{2k}$ and $\det H_n = \prod_{k=0}^n \det Q_{2k}$ for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$ and, furthermore, $\text{rank } H_{\alpha \triangleright n} = \sum_{k=0}^n \text{rank } Q_{2k+1}$ and $\det H_{\alpha \triangleright n} = \prod_{k=0}^n \det Q_{2k+1}$ for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$. Using Theorem 9.15, we obtain the asserted equations. \square

Now we characterize the membership of a sequence from $\mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$ to the classes $\mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$, $\mathcal{K}_{q, \kappa, \alpha}^{\geq}$, and $\mathcal{K}_{q, \kappa, \alpha}^{\geq, \text{cd}}$ in terms of the sequence of its α -S-transforms.

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Proposition 9.17. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq}$. For all $k \in \mathbb{Z}_{0,m}$ denote by $(s_j^{[k,\alpha]})_{j=0}^{m-k}$ the k -th α -S-transform of $(s_j)_{j=0}^m$. In view of (9.10) and (9.11), then the following statements are equivalent:*

- (i) *The sequence $(s_j)_{j=0}^m$ belongs to $\mathcal{K}_{q,m,\alpha}^{\geq,e}$.*
- (ii) *$\mathcal{N}(s_0^{[k,\alpha]}) \subseteq \mathcal{N}(s_{m-k}^{[k,\alpha]})$ for all $k \in \mathbb{Z}_{0,m-1}$ in the case $m \geq 1$.*
- (iii) *$\epsilon_{m-k,k} = 0_{q \times q}$ for all $k \in \mathbb{Z}_{0,m-1}$ in the case $m \geq 1$.*
- (iv) *$\rho_{m,m} = 0_{q \times q}$.*

Proof. In the case $m = 0$ we see from $\mathcal{K}_{q,0,\alpha}^{\geq,e} = \mathcal{K}_{q,0,\alpha}^{\geq}$ and (9.11) that both conditions (i) and (iv) are fulfilled. Furthermore, the conditions (ii) and (iii) are empty in this case, and hence, all four conditions (i), (ii), (iii), and (iv) are equivalent.

Now we consider the case $m \geq 1$.

“(i) \Rightarrow (ii)”: Let $k \in \mathbb{Z}_{0,m-1}$. Because of (i) and Theorem 8.10(b), we have $(s_j^{[k,\alpha]})_{j=0}^{m-k} \in \mathcal{K}_{q,m-k,\alpha}^{\geq,e}$, which, in view of Proposition 3.8(a), implies $(s_j^{[k,\alpha]})_{j=0}^{m-k} \in \mathcal{D}_{q \times q, m-k}$. Taking into account Definition 3.3, we obtain then $\mathcal{N}(s_0^{[k,\alpha]}) \subseteq \mathcal{N}(s_{m-k}^{[k,\alpha]})$. Hence, (ii) is fulfilled.

“(ii) \Rightarrow (iii)”: Let $k \in \mathbb{Z}_{0,m-1}$. Because of $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq}$ and Theorem 8.10(a), we have $(s_j^{[k,\alpha]})_{j=0}^{m-k} \in \mathcal{K}_{q,m-k,\alpha}^{\geq}$, which, in view of Lemma 2.3(a), implies $s_j^{[k,\alpha]} \in \mathbb{C}_H^{q \times q}$ for all $j \in \mathbb{Z}_{0,m-k}$. Using (ii), we obtain then $\mathcal{R}(s_0^{[k,\alpha]}) \subseteq \mathcal{R}(s_{m-k}^{[k,\alpha]})$. Taking into account (9.10) and parts (c) and (b) of Remark A.1, we get $\epsilon_{m-k,k} = 0_{q \times q}$. Hence (iii) is fulfilled.

“(iii) \Rightarrow (iv)”: Use (9.11).

“(iv) \Rightarrow (i)”: From Theorem 9.12 and (iv) we see that $Q_j = s_0^{[j,\alpha]}$ holds true for all $j \in \mathbb{Z}_{0,m}$, where $(Q_j)_{j=0}^m$ denotes the right α -Stieltjes parametrization of $(s_j)_{j=0}^m$. Because of Remark 8.5, we obtain $\mathcal{N}(s_0^{[m-1,\alpha]}) \subseteq \mathcal{N}(s_0^{[m,\alpha]})$. Thus, $\mathcal{N}(Q_{m-1}) \subseteq \mathcal{N}(Q_m)$ which, in view of $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq}$, Definition 2.8, and Lemmas 2.5 and 2.6, implies (i). \square

Proposition 9.18. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq}$. For all $k \in \mathbb{Z}_{0,\kappa}$ denote by $(s_j^{[k,\alpha]})_{j=0}^{\kappa-k}$ the k -th α -S-transform of $(s_j)_{j=0}^\kappa$. Then the following statements are equivalent:*

- (i) *The sequence $(s_j)_{j=0}^\kappa$ belongs to $\mathcal{K}_{q,\kappa,\alpha}^{\geq}$.*
- (ii) *For all $k \in \mathbb{Z}_{0,\kappa}$, the matrix $s_0^{[k,\alpha]}$ is non-singular.*

If (i) is fulfilled, then $s_0^{[k,\alpha]} \in \mathbb{C}_{>}^{q \times q}$ for all $k \in \mathbb{Z}_{0,\kappa}$ and, furthermore, $\epsilon_{j,k} = 0_{q \times q}$ for all $k \in \mathbb{Z}_{0,\kappa}$ and all $j \in \mathbb{Z}_{0,\kappa-k}$ and $\rho_{l,m} = 0_{q \times q}$ for all $m \in \mathbb{Z}_{0,\kappa}$ and all $l \in \mathbb{Z}_{m,\kappa}$.

Proof. “(i) \Rightarrow (ii)”: Because of (i) and Proposition 3.8(d), we have $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. Thus, Theorem 9.15 implies $s_0^{[j,\alpha]} = Q_j$ for all $j \in \mathbb{Z}_{0,\kappa}$, where $(Q_j)_{j=0}^\kappa$ denotes the right

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α -Stieltjes parametrization of $(s_j)_{j=0}^\kappa$. From (i) and Theorem 2.11(c) we obtain furthermore $Q_j \in \mathbb{C}_{>}^{q \times q}$ and hence $\det Q_j \neq 0$ for all $j \in \mathbb{Z}_{0,\kappa}$. Thus, (ii) is fulfilled and $s_0^{[k,\alpha]} \in \mathbb{C}_{>}^{q \times q}$ for all $k \in \mathbb{Z}_{0,\kappa}$.

“(ii) \Rightarrow (i)”: Because of (ii) and (9.10), we have $\epsilon_{j,k} = 0_{q \times q}$ for all $k \in \mathbb{Z}_{0,\kappa}$ and all $j \in \mathbb{Z}_{0,\kappa-k}$ which, in view of (9.11), implies $\rho_{l,m} = 0_{q \times q}$ for all $m \in \mathbb{Z}_{0,\kappa}$ and all $l \in \mathbb{Z}_{m,\kappa}$. From Remarks 9.7, 9.8, and 2.1, Corollaries 9.13 and 9.14, and (ii) we obtain then $\det H_n = \prod_{k=0}^n \det s_0^{[2k,\alpha]} \neq 0$ for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$ and $\det(-\alpha H_n + K_n) = \prod_{k=0}^n \det s_0^{[2k+1,\alpha]} \neq 0$ for all $n \in \mathbb{N}_0$ with $2n+1 \leq \kappa$, which, because of $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^\geq$, implies (i). \square

Proposition 9.19. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^\geq$. Denote by $(s_j^{[m,\alpha]})_{j=0}^0$ the m -th α -S-transform of $(s_j)_{j=0}^m$. Then $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^\geq$ if and only if $\det s_0^{[m,\alpha]} \neq 0$.*

Proof. Use Proposition 9.18 and Remark 8.5. \square

Proposition 9.20. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, and let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^\geq$. Denote by $(s_j^{[m,\alpha]})_{j=0}^0$ the m -th α -S-transform of $(s_j)_{j=0}^m$. Then the following statements are equivalent:*

(i) *The sequence $(s_j)_{j=0}^m$ belongs to $\mathcal{K}_{q,m,\alpha}^{\geq, \text{cd}}$.*

(ii) $s_0^{[m,\alpha]} + \rho_{m,m} = 0_{q \times q}$.

If (i) is fulfilled, then $s_0^{[m,\alpha]} = 0_{q \times q}$, $\epsilon_{j,k} = 0_{q \times q}$ for all $k \in \mathbb{Z}_{0,m}$ and all $j \in \mathbb{Z}_{0,m-k}$, and $\rho_{l,n} = 0_{q \times q}$ for all $n \in \mathbb{Z}_{0,m}$ and all $l \in \mathbb{Z}_{n,m}$.

Proof. “(i) \Rightarrow (ii)”: Because of (i) and Proposition 3.8(d), we have $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq, \text{e}}$. Thus, Lemma 9.4 yields $\epsilon_{j,k} = 0_{q \times q}$ for all $k \in \mathbb{Z}_{0,m}$ and all $j \in \mathbb{Z}_{0,m-k}$, and $\rho_{l,n} = 0_{q \times q}$ for all $n \in \mathbb{Z}_{0,m}$ and all $l \in \mathbb{Z}_{n,m}$, whereas Theorem 9.15 yields $s_0^{[m,\alpha]} = Q_m$, where $(Q_j)_{j=0}^m$ denotes the right α -Stieltjes parametrization of $(s_j)_{j=0}^m$. From (i) and [21, Proposition 5.3] we obtain furthermore $Q_m = 0_{q \times q}$. Hence, (ii) is fulfilled.

“(ii) \Rightarrow (i)”: Because of $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^\geq$, the application of Theorem 9.12 yields $Q_m = s_0^{[m,\alpha]} + \rho_{m,m}$, which, in view of (ii), implies $Q_m = 0_{q \times q}$. Taking into account $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^\geq$, from [21, Proposition 5.3] we obtain then (i). \square

The following example shows that in the situation of Proposition 9.20 condition (ii) cannot be weakened by replacing it by the condition $s_0^{[m,\alpha]} = 0_{q \times q}$.

Example 9.21. Let $\alpha \in \mathbb{R}$, let $s_0 := 0_{q \times q}$, and let $s_1 := I_q$. According to (1.3), then $s_{\alpha \triangleright 0} = I_q$. Because of (1.1) and (2.15), we have then $H_0 = s_0 = 0_{q \times q} \in \mathbb{C}_{>}^{q \times q}$ and $H_{\alpha \triangleright 0} = s_{\alpha \triangleright 0} = I_q \in \mathbb{C}_{>}^{q \times q}$, which, in view of (1.5), implies $(s_j)_{j=0}^1 \in \mathcal{K}_{q,1,\alpha}^\geq$. By Definition 7.1, furthermore $s_0^{[1,\alpha]} = -s_0 s_1^{[\sharp,\alpha]} s_0 = -s_0 s_1^{[\sharp,\alpha]} \cdot 0_{q \times q} = 0_{q \times q}$. Because of (2.16) and (2.11), moreover $L_{\alpha \triangleright 0} = s_{\alpha \triangleright 0} = I_q \neq 0_{q \times q}$ which, in view of (2.4) and (2.2), implies $(s_j)_{j=0}^1 \notin \mathcal{K}_{q,1,\alpha}^{\geq, \text{cd}}$. Hence, $(s_j)_{j=0}^1 \in \mathcal{K}_{q,1,\alpha}^\geq \setminus \mathcal{K}_{q,1,\alpha}^{\geq, \text{cd}}$ although $s_0^{[1,\alpha]} = 0_{q \times q}$.

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Proposition 9.22. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N} \cup \{+\infty\}$, let $m \in \mathbb{Z}_{0,\kappa-1}$, and let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^\geq$. For all $k \in \mathbb{Z}_{m,\kappa}$ denote by $(s_j^{[k,\alpha]})_{j=0}^\kappa$ the k -th α -S-transform of $(s_j)_{j=0}^\kappa$. Then the following statements are equivalent:*

(i) *The sequence $(s_j)_{j=0}^\kappa$ belongs to $\mathcal{K}_{q,\kappa,\alpha}^{\geq,\text{cd},m}$.*

(ii) $s_0^{[m,\alpha]} = 0_{q \times q}$.

If (i) is fulfilled, then $s_j^{[k,\alpha]} = 0_{q \times q}$ for all $k \in \mathbb{Z}_{m+1,\kappa}$ and all $j \in \mathbb{Z}_{0,\kappa-k}$, and $\epsilon_{j,k} = 0_{q \times q}$ for all $k \in \mathbb{Z}_{0,\kappa-1}$ and all $j \in \mathbb{Z}_{0,\kappa-1-k}$ and, furthermore, $\rho_{l,n} = 0_{q \times q}$ for all $n \in \mathbb{Z}_{0,\kappa-1}$ and all $l \in \mathbb{Z}_{n,\kappa-1}$.

Proof. Because of $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^\geq$ and $m \in \mathbb{Z}_{0,\kappa-1}$, we have $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^\geq$ and, in view of $\kappa \in \mathbb{N} \cup \{+\infty\}$ and Remark 9.5, furthermore $\epsilon_{j,k} = 0_{q \times q}$ for all $k \in \mathbb{Z}_{0,\kappa-1}$ and all $j \in \mathbb{Z}_{0,\kappa-1-k}$ and $\rho_{l,n} = 0_{q \times q}$ for all $n \in \mathbb{Z}_{0,\kappa-1}$ and all $l \in \mathbb{Z}_{n,\kappa-1}$. In particular, $\rho_{m,m} = 0_{q \times q}$.

“(i) \Rightarrow (ii)”: From (i) and (2.5) we get $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,\text{cd}}$. Thus, Proposition 9.20 yields (ii). From Remark 8.5 we obtain furthermore $\text{rank } s_j^{[k,\alpha]} \leq \text{rank } s_0^{[m,\alpha]} = 0$ and thus $s_j^{[k,\alpha]} = 0_{q \times q}$ for all $k \in \mathbb{Z}_{m+1,\kappa}$ and all $j \in \mathbb{Z}_{0,\kappa-k}$.

“(ii) \Rightarrow (i)”: Because of (ii) and $\rho_{m,m} = 0_{q \times q}$, we have $s_0^{[m,\alpha]} + \rho_{m,m} = 0_{q \times q}$. The application of Proposition 9.20 yields then $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,\text{cd}}$, which, in view of (2.5), implies (i). \square

The following example shows that in the situation of Proposition 9.22 we can from condition (i) in general not conclude that $s_j^{[m,\alpha]} = 0_{q \times q}$ for all $j \in \mathbb{Z}_{1,\kappa-m}$.

Example 9.23. Let $\alpha \in \mathbb{R}$, let $s_0 := 0_{q \times q}$, and let $s_1 := I_q$. According to Example 9.21, then $(s_j)_{j=0}^1 \in \mathcal{K}_{q,1,\alpha}^\geq$. In particular, $(s_j)_{j=0}^0 \in \mathcal{K}_{q,0,\alpha}^\geq$. Because of (2.11), we have $L_0 = s_0 = 0_{q \times q}$. In view of (2.3) and (2.2), thus $(s_j)_{j=0}^0 \in \mathcal{K}_{q,0,\alpha}^{\geq,\text{cd}}$ and, according to (2.5), we see that $(s_j)_{j=0}^1 \in \mathcal{K}_{q,1,\alpha}^{\geq,\text{cd},0}$. Taking into account Definition 8.1, moreover $s_1^{[0,\alpha]} = s_1 = I_q$. Hence, $s_1^{[0,\alpha]} \neq 0_{q \times q}$ although $(s_j)_{j=0}^1 \in \mathcal{K}_{q,1,\alpha}^{\geq,\text{cd},0}$.

Proposition 9.24. *Let $\alpha \in \mathbb{R}$ and let $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty,\alpha}^\geq$. For all $k \in \mathbb{N}_0$, denote by $(s_j^{[k,\alpha]})_{j=0}^\infty$ the k -th α -S-transform of $(s_j)_{j=0}^\infty$. Then the following statements are equivalent:*

(i) *The sequence $(s_j)_{j=0}^\infty$ belongs to $\mathcal{K}_{q,\infty,\alpha}^{\geq,\text{cd}}$.*

(ii) *There exists an $m \in \mathbb{N}_0$ such that $s_0^{[m,\alpha]} = 0_{q \times q}$.*

If (ii) is fulfilled and if $m \in \mathbb{N}_0$ is such that $s_0^{[m,\alpha]} = 0_{q \times q}$, then $s_j^{[k,\alpha]} = 0_{q \times q}$ for all $j \in \mathbb{N}_0$ and all $k \in \mathbb{Z}_{m,\infty}$, $\epsilon_{j,k} = 0_{q \times q}$ for all $j, k \in \mathbb{N}_0$, and $\rho_{l,n} = 0_{q \times q}$ for all $n \in \mathbb{N}_0$ and all $l \in \mathbb{Z}_{n,\infty}$.

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Proof. The equivalence of (i) and (ii) is a consequence of (2.6) and Proposition 9.22. Now suppose that (ii) is fulfilled and let $m \in \mathbb{N}_0$ be such that $s_0^{[m,\alpha]} = 0_{q \times q}$. Theorem 8.10(a) yields then $(s_j^{[m,\alpha]})_{j=0}^\infty \in \mathcal{K}_{q,\infty,\alpha}^\geq$, which, in view of Lemma 2.3(c), implies $\mathcal{R}(s_j^{[m,\alpha]}) \subseteq \mathcal{R}(s_0^{[m,\alpha]}) = \{0_{q \times 1}\}$ and hence $s_j^{[m,\alpha]} = 0_{q \times q}$ for all $j \in \mathbb{N}_0$. From Remark 8.5 we obtain furthermore $\text{rank } s_j^{[k,\alpha]} \leq \text{rank } s_0^{[m,\alpha]} = 0$ and thus $s_j^{[k,\alpha]} = 0_{q \times q}$ for all $k \in \mathbb{Z}_{m+1,\infty}$ and all $j \in \mathbb{N}_0$. Because of $(s_j)_{j=0}^\infty \in \mathcal{K}_{q,\infty,\alpha}^\geq = \mathcal{K}_{q,\infty,\alpha}^{\geq,e}$ and Lemma 9.4, we have $\epsilon_{j,k} = 0_{q \times q}$ for all $j, k \in \mathbb{N}_0$ and $\rho_{l,n} = 0_{q \times q}$ for all $n \in \mathbb{N}_0$ and all $l \in \mathbb{Z}_{n,\infty}$, which completes the proof. \square

The following two Theorems 9.25 and 9.26 are further main results of this paper. They describe the connection between the right α -Stieltjes parametrizations of a sequence from $\mathcal{K}_{q,m,\alpha}^\geq$ and $\mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$ and its k -th α -S-transform.

Theorem 9.25. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, let $(s_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^\geq$, and let $k \in \mathbb{Z}_{0,m}$. Denote by $(Q_j)_{j=0}^m$ and $(T_j)_{j=0}^{m-k}$ the right α -Stieltjes parametrizations of $(s_j)_{j=0}^m$ and $(t_j)_{j=0}^{m-k}$, respectively, where $(t_j)_{j=0}^{m-k}$ denotes the k -th α -S-transform of $(s_j)_{j=0}^m$. Then*

$$T_{m-k} = \begin{cases} Q_m & \text{if } k = 0 \\ Q_m - \sum_{r=0}^{k-1} \epsilon_{m-r,r} & \text{if } k \geq 1 \end{cases}$$

and, in the case $k < m$, furthermore $T_j = Q_{k+j}$ for all $j \in \mathbb{Z}_{0,m-k-1}$.

Proof. From Theorem 8.10(a) we obtain $(t_j)_{j=0}^{m-k} \in \mathcal{K}_{q,m-k,\alpha}^\geq$. For all $l \in \mathbb{Z}_{0,m-k}$, denote by $(t_j^{[l,\alpha]})_{j=0}^{m-k-l}$ the l -th α -S-transform of $(t_j)_{j=0}^{m-k}$. According to Remark 8.3, we have $s_j^{[k+l,\alpha]} = t_j^{[l,\alpha]}$ for all $l \in \mathbb{Z}_{0,m-k}$ and all $j \in \mathbb{Z}_{0,m-k-l}$. The application of Theorem 9.12 yields $Q_m = s_0^{[m,\alpha]} + \rho_{m,m}$.

We first consider the case $k = m$. The application of Theorem 9.12 to the sequence $(t_j)_{j=0}^{m-k}$ yields, in view of $m - k = 0$ and (9.11), then $T_{m-k} = t_0^{[m-k,\alpha]} + 0_{q \times q}$. Hence, $T_{m-k} = t_0^{[m-k,\alpha]} = s_0^{[m,\alpha]} = Q_m - \rho_{m,m} = Q_m - \rho_{m,k}$. Taking into account (9.11), this implies $T_{m-k} = Q_m$ in the case $k = 0$ and $T_{m-k} = Q_m - \sum_{r=0}^{k-1} \epsilon_{m-r,r}$ in the case $k \geq 1$ which completes the proof in the case $k = m$.

Now suppose $k < m$. Then $m \geq 1$ and $m - k \geq 1$. The application of Theorem 9.12 to the sequence $(t_j)_{j=0}^{m-k}$ yields, in view of (9.11) and (9.10), thus

$$T_{m-k} = t_0^{[m-k,\alpha]} + \sum_{l=0}^{m-k-1} \left[t_{m-k-l}^{[l,\alpha]} - t_0^{[l,\alpha]} (t_0^{[l,\alpha]})^\dagger t_{m-k-l}^{[l,\alpha]} (t_0^{[l,\alpha]})^\dagger t_0^{[l,\alpha]} \right]$$

and $T_j = t_0^{[j,\alpha]}$ for all $j \in \mathbb{Z}_{0,m-k-1}$. Furthermore, $m \geq 1$ and Theorem 9.12 yield

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$Q_j = s_0^{[j, \alpha]}$ for all $j \in \mathbb{Z}_{0, m-1}$. Taking into account (9.10) and (9.11), we have thus

$$\begin{aligned} T_{m-k} &= s_0^{[m, \alpha]} + \sum_{l=0}^{m-k-1} \left[s_{m-k-l}^{[k+l, \alpha]} - s_0^{[k+l, \alpha]} (s_0^{[k+l, \alpha]})^\dagger s_{m-k-l}^{[k+l, \alpha]} (s_0^{[k+l, \alpha]})^\dagger s_0^{[k+l, \alpha]} \right] \\ &= Q_m - \rho_{m, m} + \sum_{l=0}^{m-k-1} \epsilon_{m-k-l, k+l} = Q_m - \sum_{r=0}^{m-1} \epsilon_{m-r, r} + \sum_{r=k}^{m-1} \epsilon_{m-r, r} \end{aligned}$$

and $T_j = s_0^{[k+j, \alpha]} = Q_{k+j}$ for all $j \in \mathbb{Z}_{0, m-k-1}$. In particular, $T_{m-k} = Q_m$ in the case $k = 0$ and $T_{m-k} = Q_m - \sum_{r=0}^{k-1} \epsilon_{m-r, r}$ in the case $k \geq 1$, which completes the proof. \square

Now we consider for a sequence $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$ the same task as it was done in Theorem 9.25 for the case of α -Stieltjes non-negative definite sequences.

Theorem 9.26. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(s_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$, and let $k \in \mathbb{Z}_{0, \kappa}$. Denote by $(t_j)_{j=0}^{\kappa-k}$ the k -th α -S-transform of $(s_j)_{j=0}^\kappa$ and by $(Q_j)_{j=0}^\kappa$ the right α -Stieltjes parametrization of $(s_j)_{j=0}^\kappa$. Then $(Q_{k+j})_{j=0}^{\kappa-k}$ is exactly the right α -Stieltjes parametrization of $(t_j)_{j=0}^{\kappa-k}$.*

Proof. Let $l \in \mathbb{Z}_{0, \kappa-k}$. Then $k+l \in \mathbb{Z}_{0, \kappa}$, and the application of Theorem 9.15 yields $s_0^{[k+l, \alpha]} = Q_{k+l}$. According to Remark 8.3, we have $s_0^{[k+l, \alpha]} = t_0^{[l, \alpha]}$, where $(t_j^{[l, \alpha]})_{j=0}^{\kappa-k-l}$ denotes the l -th α -S-transform of $(t_j)_{j=0}^{\kappa-k}$. From Theorem 8.10(b) we obtain $(t_j)_{j=0}^{\kappa-k} \in \mathcal{K}_{q, \kappa-k, \alpha}^{\geq, e}$. The application of Theorem 9.15 to the sequence $(t_j)_{j=0}^{\kappa-k}$ yields then $t_0^{[l, \alpha]} = T_l$, where $(T_j)_{j=0}^{\kappa-k}$ is the right α -Stieltjes parametrization of $(t_j)_{j=0}^{\kappa-k}$. Thus, we have finally $Q_{k+j} = s_0^{[k+j, \alpha]} = t_0^{[j, \alpha]} = T_j$ for all $j \in \mathbb{Z}_{0, \kappa-k}$, which completes the proof. \square

Theorems 9.25 and 9.26 indicate that against to the background of our Schur-type algorithm the right α -Stieltjes parametrization can be interpreted as a Schur-type parametrization.

10. Recovering the original sequence from its first α -Schur-transform and its first matrix

The considerations in Section 7 suggest the study of a natural inverse problem associated with the first α -S-transform of a (finite or infinite) sequence of complex $p \times q$ matrices. The main theme of this section is the treatment of this inverse problem, which will be explained below in more detail. It should be mentioned that a similar task was treated in [25, Section 10] against to the background of the Schur-type algorithm studied there. The first study of inverse problems of this kind goes back to the papers [4], where the inverse problem associated with the Schur-Potapov algorithm for strict $p \times q$ Schur sequences was handled.

Let $\alpha \in \mathbb{C}$ and $\kappa \in \mathbb{N} \cup \{+\infty\}$. For each sequence $(s_j)_{j=0}^\kappa$ of complex $p \times q$ matrices the first α -S-transform $(s_j^{[1, \alpha]})_{j=0}^{\kappa-1}$ is given by Definition 7.1. Conversely, we consider the

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question: If the first α -S-transform $(s_j^{[1,\alpha]})_{j=0}^{\kappa-1}$ and the matrix s_0 are known, how one can recover the original sequence $(s_j)_{j=0}^\kappa$. If $m \in \mathbb{N}$ and if $(s_j)_{j=0}^m$ belongs to $\mathcal{K}_{q,m,\alpha}^\geq$ then Theorem 9.12, Definition 8.1, and Remark 2.9 yield such a possibility to express $(s_j)_{j=0}^m$ by $(s_j^{[1,\alpha]})_{j=0}^{m-1}$ and s_0 . In view of the definition of the right α -Stieltjes parametrization and the formulas (2.11), (2.13), (2.14), and (2.16), this way of computation is not very comfortable. That's why it seems to be more advantageous to construct a recursive procedure to recover the original sequence $(s_j)_{j=0}^m$ from its first α -S-transform and the matrix s_0 . To realize this aim, first we introduce the central construction of this section.

Definition 10.1. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(t_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices, and let A be a complex $p \times q$ matrix. The sequence $(t_j^{[-1,\alpha,A]})_{j=0}^{\kappa+1}$ recursively defined by

$$t_0^{[-1,\alpha,A]} := A \quad \text{and} \quad t_j^{[-1,\alpha,A]} := \alpha^j A + \sum_{l=1}^j \alpha^{j-l} A A^\dagger \left[\sum_{k=0}^{l-1} t_{l-k-1} A^\dagger (t_k^{[-1,\alpha,A]})^{[+,\alpha]} \right]$$

for all $j \in \mathbb{Z}_{1,\kappa+1}$ is called the *first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$* .

Remark 10.2. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(t_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices, and let A be a complex $p \times q$ matrix. Denote by $(s_j)_{j=0}^{\kappa+1}$ the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$. In view of Definition 10.1, one can easily see that, for all $m \in \mathbb{Z}_{0,\kappa}$, the sequence $(s_j)_{j=0}^{m+1}$ depends only on the matrices A and t_0, t_1, \dots, t_m and is hence exactly the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^m, A]$.

The following observation expresses an essential feature of our construction.

Remark 10.3. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(t_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices, and let A be a complex $p \times q$ matrix. Denote by $(s_j)_{j=0}^{\kappa+1}$ the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$. From Definitions 10.1 and 3.3 we easily see then that $(s_j)_{j=0}^{\kappa+1}$ belongs to $\mathcal{D}_{p \times q, \kappa+1}$.

Lemma 10.4. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(t_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices, and let A be a complex $p \times q$ matrix. Denote by $(s_j)_{j=0}^{\kappa+1}$ the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$ and by $(s_j^{[+,\alpha]})_{j=0}^{\kappa+1}$ the $[+,\alpha]$ -transform of $(s_j)_{j=0}^{\kappa+1}$. Then $s_0 = A$ and $s_j = \alpha s_{j-1} + A A^\dagger \sum_{k=0}^{j-1} t_{j-1-k} A^\dagger s_k^{[+,\alpha]}$ for all $j \in \mathbb{Z}_{1,\kappa+1}$.

Proof. Using Definition 10.1, we obtain $s_0 = A$ and $s_1 = \alpha A + A A^\dagger t_0 A^\dagger s_0^{[+,\alpha]} = \alpha s_0 +$

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$AA^\dagger t_0 A^\dagger s_0^{[+, \alpha]}$ and, in the case $\kappa \geq 1$, for all $j \in \mathbb{Z}_{2, \kappa+1}$, furthermore

$$\begin{aligned}
& \alpha s_{j-1} + AA^\dagger \sum_{k=0}^{j-1} t_{j-1-k} A^\dagger s_k^{[+, \alpha]} \\
&= \alpha \left[\alpha^{j-1} A + \sum_{l=1}^{j-1} \alpha^{j-1-l} AA^\dagger \left(\sum_{k=0}^{l-1} t_{l-k-1} A^\dagger s_k^{[+, \alpha]} \right) \right] + AA^\dagger \sum_{k=0}^{j-1} t_{j-1-k} A^\dagger s_k^{[+, \alpha]} \\
&= \alpha^j A + \sum_{l=1}^{j-1} \alpha^{j-l} AA^\dagger \left(\sum_{k=0}^{l-1} t_{l-k-1} A^\dagger s_k^{[+, \alpha]} \right) + AA^\dagger \sum_{k=0}^{j-1} t_{j-k-1} A^\dagger s_k^{[+, \alpha]} \\
&= \alpha^j A + \sum_{l=1}^j \alpha^{j-l} AA^\dagger \left(\sum_{k=0}^{l-1} t_{l-k-1} A^\dagger s_k^{[+, \alpha]} \right) = s_j,
\end{aligned}$$

which completes the proof. \square

For a given number $\alpha \in \mathbb{C}$, a given $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and a given sequence $(t_j)_{j=0}^\kappa$ of complex $p \times q$ matrices, we want to determine a complex $p \times q$ matrix A such that the sequence $(t_j)_{j=0}^\kappa$ turns out to be exactly the first α -S-transform of the first inverse α -S-transform $(t_j^{[-1, \alpha, A]})_{j=0}^{\kappa+1}$ corresponding to $[(t_j)_{j=0}^\kappa, A]$. To realize this aim, we still need a little preparation.

Lemma 10.5. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(t_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices, and let A be a complex $p \times q$ matrix. Denote by $(s_j)_{j=0}^{\kappa+1}$ the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$ and let the sequence $(r_j)_{j=0}^{\kappa+1}$ be given by $r_j := s_j^{[\sharp, \alpha]}$ for all $j \in \mathbb{Z}_{0, \kappa+1}$. For all $j \in \mathbb{Z}_{0, \kappa+1}$, then*

$$r_j = \begin{cases} A^\dagger & \text{if } j = 0 \\ -A^\dagger t_{j-1} A^\dagger & \text{if } j \geq 1 \end{cases} \quad \text{and} \quad r_j^\sharp = s_j^{[+, \alpha]}.$$

Proof. Let the sequence $(u_j)_{j=0}^{\kappa+1}$ be given by $u_0 := A^\dagger$ and $u_j := -A^\dagger t_{j-1} A^\dagger$ for all $j \in \mathbb{Z}_{1, \kappa+1}$ and let the sequence $(v_j)_{j=0}^{\kappa+1}$ be given by $v_j := s_j^{[+, \alpha]}$ for all $j \in \mathbb{Z}_{0, \kappa+1}$. Using (4.1), Definition 10.1, Remark A.1(a), and Definition 3.1, we get then $v_0 = s_0 = A = (A^\dagger)^\dagger = u_0^\dagger = u_0^\sharp$ and, taking additionally this and Definition 4.1 into account, furthermore

$$\begin{aligned}
v_1 &= -\alpha s_0 + s_1 = -\alpha A + \alpha^1 A + \sum_{l=1}^1 \alpha^{1-l} AA^\dagger \left(\sum_{k=0}^{l-1} t_{l-k-1} A^\dagger v_k \right) \\
&= -(A^\dagger)^\dagger (-A^\dagger t_{1-1} A^\dagger) v_0 = -u_0^\dagger u_1 u_0^\sharp = u_1^\sharp.
\end{aligned}$$

Consequently, if $\kappa = 0$, for all $j \in \mathbb{Z}_{0, \kappa+1}$, then

$$v_j = u_j^\sharp. \tag{10.1}$$

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Let us consider the case $\kappa \geq 1$. Then we have already proved that there exists a number $m \in \mathbb{Z}_{1,\kappa}$ such that (10.1) is satisfied for all $j \in \mathbb{Z}_{0,m}$. Because of Definitions 4.1 and 10.1, (10.1), Remark A.1(a), and Definition 3.1, we conclude that

$$\begin{aligned}
v_{m+1} &= -\alpha s_m + s_{m+1} \\
&= -\alpha \left[\alpha^m A + \sum_{l=1}^m \alpha^{m-l} A A^\dagger \left(\sum_{k=0}^{l-1} t_{l-k-1} A^\dagger v_k \right) \right] \\
&\quad + \left[\alpha^{m+1} A + \sum_{l=1}^{m+1} \alpha^{(m+1)-l} A A^\dagger \left(\sum_{k=0}^{l-1} t_{l-k-1} A^\dagger v_k \right) \right] \\
&= -\sum_{l=1}^m \alpha^{m-l+1} A A^\dagger \left(\sum_{k=0}^{l-1} t_{l-k-1} A^\dagger u_k^\# \right) + \sum_{l=1}^{m+1} \alpha^{m-l+1} A A^\dagger \left(\sum_{k=0}^{l-1} t_{l-k-1} A^\dagger u_k^\# \right) \\
&= A A^\dagger \sum_{k=0}^m t_{m-k} A^\dagger u_k^\# = -(A^\dagger)^\dagger \sum_{k=0}^m (-A^\dagger t_{m-k} A^\dagger) u_k^\# = -u_0^\dagger \sum_{k=0}^m u_{m-k+1} u_k^\# = u_{m+1}^\#.
\end{aligned}$$

Hence, (10.1) is proved inductively for all $j \in \mathbb{Z}_{0,\kappa+1}$. In view of Definition 3.3, the sequence $(u_j)_{j=0}^{\kappa+1}$ obviously belongs to $\mathcal{D}_{p \times q, \kappa+1}$. Taking additionally into account (10.1), then [26, Corollary 4.22] yields $v_j^\# = u_j$ for all $j \in \mathbb{Z}_{0,\kappa+1}$. In view of (4.4), we have thus $r_j = s_j^{[\sharp, \alpha]} = v_j^\# = u_j$ for all $j \in \mathbb{Z}_{0,\kappa+1}$. Using this and (10.1) then $r_j^\# = u_j^\# = v_j = s_j^{[+, \alpha]}$ follows for all $j \in \mathbb{Z}_{0,\kappa+1}$. \square

Lemma 10.6. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(t_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices, and let A be a complex $p \times q$ matrix. Denote by $(s_j)_{j=0}^{\kappa+1}$ the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$ and by $(s_j^{[1, \alpha]})_{j=0}^\kappa$ the first α -S-transform of $(s_j)_{j=0}^{\kappa+1}$. Then:*

(a) $s_j^{[1, \alpha]} = A A^\dagger t_j A^\dagger A$ for all $j \in \mathbb{Z}_{0,\kappa}$.

(b) If $\mathcal{N}(A) \subseteq \bigcap_{j=0}^\kappa \mathcal{N}(t_j)$ and $\bigcup_{j=0}^\kappa \mathcal{R}(t_j) \subseteq \mathcal{R}(A)$, then $s_j^{[1, \alpha]} = t_j$ for all $j \in \mathbb{Z}_{0,\kappa}$.

Proof. Using Definitions 7.1 and 10.1 and Lemma 10.5, we obtain

$$s_j^{[1, \alpha]} = -s_0 s_{j+1}^{[\sharp, \alpha]} s_0 = -A(-A^\dagger t_{j+1-1} A^\dagger) A = A A^\dagger t_j A^\dagger A$$

for all $j \in \mathbb{Z}_{0,\kappa}$. Thus part (a) is proved. Part (b) is an immediate consequence of (a) and parts (c) and (b) of Remark A.1. \square

Remark 10.7. Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(t_j)_{j=0}^\kappa \in \mathcal{D}_{p \times q, \kappa}$, and let $A \in \mathbb{C}^{p \times q}$ be such that $\mathcal{N}(A) \subseteq \mathcal{N}(t_0)$ and $\mathcal{R}(t_0) \subseteq \mathcal{R}(A)$. In view of Definition 3.3, then $\mathcal{N}(A) \subseteq \bigcap_{j=0}^\kappa \mathcal{N}(t_j)$ and $\bigcup_{j=0}^\kappa \mathcal{R}(t_j) \subseteq \mathcal{R}(A)$.

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Proposition 10.8. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(t_j)_{j=0}^\kappa \in \mathcal{D}_{p \times q, \kappa}$, and let $A \in \mathbb{C}^{p \times q}$ be such that $\mathcal{N}(A) \subseteq \mathcal{N}(t_0)$ and $\mathcal{R}(t_0) \subseteq \mathcal{R}(A)$. Denote by $(s_j)_{j=0}^{\kappa+1}$ the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$. Then $(t_j)_{j=0}^\kappa$ is exactly the first α -S-transform of $(s_j)_{j=0}^{\kappa+1}$.*

Proof. Use Remark 10.7 and Lemma 10.6(b). \square

Our next considerations can be sketched as follows. Let $\alpha \in \mathbb{C}$, $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and $(s_j)_{j=0}^{\kappa+1} \in \mathcal{D}_{p \times q, \kappa+1}$. Then we want to recover $(s_j)_{j=0}^{\kappa+1}$ from its first α -S-transform and its first matrix s_0 . For this reason, we still need a little preparation.

Lemma 10.9. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^{\kappa+1}$ be a sequence of complex $p \times q$ matrices. Denote by $(t_j)_{j=0}^\kappa$ the first α -S-transform of $(s_j)_{j=0}^{\kappa+1}$ and by $(w_j)_{j=0}^{\kappa+1}$ the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, s_0]$. Then $w_j = s_0 s_0^\dagger s_j s_0^\dagger s_0$ for all $j \in \mathbb{Z}_{0, \kappa+1}$.*

Proof. Let the sequence $(r_j)_{j=0}^{\kappa+1}$ be given by $r_j := w_j^{[\sharp, \alpha]}$ for all $j \in \mathbb{Z}_{0, \kappa+1}$. Lemma 10.5 yields then $r_0 = s_0^\dagger$ and $r_j = -s_0^\dagger t_{j-1} s_0^\dagger$ for all $j \in \mathbb{Z}_{1, \kappa+1}$. Using Lemma 4.6 and Definition 3.1 we obtain $s_0^{[\sharp, \alpha]} = s_0^\dagger$. For all $j \in \mathbb{Z}_{1, \kappa+1}$ we get from Definition 7.1 furthermore $-s_0^\dagger t_{j-1} s_0^\dagger = s_0^\dagger s_0 s_j^{[\sharp, \alpha]} s_0 s_0^\dagger$. According to Lemma 4.8, the sequence $(s_j^{[\sharp, \alpha]})_{j=0}^{\kappa+1}$ belongs to $\mathcal{D}_{q \times p, \kappa+1}$. Taking additionally into account $s_0^{[\sharp, \alpha]} = s_0^\dagger$ and Definition 3.3, we obtain $\bigcup_{j=0}^{\kappa+1} \mathcal{R}(s_j^{[\sharp, \alpha]}) \subseteq \mathcal{R}(s_0^\dagger)$ and $\mathcal{N}(s_0^\dagger) \subseteq \bigcap_{j=0}^{\kappa+1} \mathcal{N}(s_j^{[\sharp, \alpha]})$ which, in view of parts (a), (c), and (b) of Remark A.1, implies $s_0^\dagger s_0 s_j^{[\sharp, \alpha]} s_0 s_0^\dagger = s_j^{[\sharp, \alpha]}$ for all $j \in \mathbb{Z}_{1, \kappa+1}$. We have, for all $j \in \mathbb{Z}_{0, \kappa+1}$, thus $r_j = s_j^{[\sharp, \alpha]}$ and hence $w_j^{[\sharp, \alpha]} = s_j^{[\sharp, \alpha]}$. The application of Lemma 4.10 then yields $w_0 w_0^\dagger w_j w_0^\dagger w_0 = s_0 s_0^\dagger s_j s_0^\dagger s_0$ for all $j \in \mathbb{Z}_{0, \kappa+1}$. According to Remark 10.3, the sequence $(w_j)_{j=0}^{\kappa+1}$ belongs to $\mathcal{D}_{p \times q, \kappa+1}$. Taking additionally into account Definition 3.3 and parts (c) and (b) of Remark A.1, we get consequently $w_0 w_0^\dagger w_j w_0^\dagger w_0 = w_j$ for all $j \in \mathbb{Z}_{0, \kappa+1}$, which completes the proof. \square

Proposition 10.10. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(s_j)_{j=0}^{\kappa+1} \in \mathcal{D}_{p \times q, \kappa+1}$. Denote by $(t_j)_{j=0}^\kappa$ the first α -S-transform of $(s_j)_{j=0}^{\kappa+1}$. Then $(s_j)_{j=0}^{\kappa+1}$ is exactly the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, s_0]$.*

Proof. Use Lemma 10.9, Definition 3.3, and parts (c) and (b) of Remark A.1. \square

Propositions 10.8 and 10.10 indicate the particular role of the class $\mathcal{D}_{p \times q, \kappa}$ of first term dominant sequences of complex $p \times q$ matrices (see Definition 3.3) in the context of this section. Roughly speaking, the aim of our following considerations can be described as follows: Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, and let $(t_j)_{j=0}^\kappa$ be a sequence which belongs to $\mathcal{K}_{q, \kappa, \alpha}^{\geq}$ or to one of its distinguished subclasses $\mathcal{K}_{q, \kappa, \alpha}^{\geq, e}$, $\mathcal{K}_{q, \kappa, \alpha}^{>}$, and $\mathcal{K}_{q, \kappa, \alpha}^{\geq, \text{cd}}$. Then we are looking for matrices $A \in \mathbb{C}^{q \times q}$ such that the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$ belongs to $\mathcal{K}_{q, \kappa+1, \alpha}^{\geq}$, $\mathcal{K}_{q, \kappa+1, \alpha}^{\geq, e}$, $\mathcal{K}_{q, \kappa+1, \alpha}^{>}$, and $\mathcal{K}_{q, \kappa+1, \alpha}^{\geq, \text{cd}}$, respectively. First we derive some formulas which express useful interrelations between essential block Hankel matrices occurring in our considerations.

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Proposition 10.11. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(t_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices, and let A be a complex $p \times q$ matrix. Denote by $(s_j)_{j=0}^{\kappa+1}$ the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$. Then:*

(a) $H_0 = A$.

(b) *Let $n \in \mathbb{N}$ with $2n - 1 \leq \kappa$. Then the matrices \mathbf{D}_n and \mathbb{D}_n given via (7.3) and (7.4) are invertible and*

$$H_n = \mathbf{D}_n^{-1} \left(\text{diag} \left[A, \left[I_n \otimes (AA^\dagger) \right] (-\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)}) \left[I_n \otimes (A^\dagger A) \right] \right] \right) \mathbb{D}_n^{-1}. \quad (10.2)$$

In particular,

$$\text{rank } H_n = \text{rank}(A) + \text{rank} \left(\left[I_n \otimes (AA^\dagger) \right] (-\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)}) \left[I_n \otimes (A^\dagger A) \right] \right) \quad (10.3)$$

and, in the case $p = q$, furthermore $\det H_n = \det(A) \det(-\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)})$.

(c) *Let $n \in \mathbb{N}_0$ with $2n \leq \kappa$. Then the matrices $\mathbf{D}_n^{[+, \alpha]}$ and $\mathbb{D}_n^{[+, \alpha]}$ given via (7.5) are invertible and*

$$H_{\alpha \triangleright n} = (\mathbf{D}_n^{[+, \alpha]})^{-1} \left[I_{n+1} \otimes (AA^\dagger) \right] H_n^{(t)} \left[I_{n+1} \otimes (A^\dagger A) \right] (\mathbb{D}_n^{[+, \alpha]})^{-1}. \quad (10.4)$$

In particular,

$$\text{rank } H_{\alpha \triangleright n} = \text{rank} \left(\left[I_{n+1} \otimes (AA^\dagger) \right] H_n^{(t)} \left[I_{n+1} \otimes (A^\dagger A) \right] \right) \quad (10.5)$$

and, in the case $p = q$, furthermore $\det H_{\alpha \triangleright n} = (\det A)(\det A)^\dagger \det H_n^{(t)}$.

Proof. First observe that Definition 10.1 yields $s_0 = A$, that Remark 10.3 yields $(s_j)_{j=0}^{\kappa+1} \in \mathcal{D}_{p \times q, \kappa+1}$, and that Lemma 10.6(a) yields $s_j^{[1, \alpha]} = AA^\dagger t_j A^\dagger A$ for all $j \in \mathbb{Z}_{0, \kappa}$, where $(s_j^{[1, \alpha]})_{j=0}^\kappa$ denotes the first α -S-transform of $(s_j)_{j=0}^{\kappa+1}$.

(a) Use (1.1) and $s_0 = A$.

(b) In view of Remark 7.15, the matrices \mathbf{D}_n and \mathbb{D}_n are invertible. Because of $(s_j)_{j=0}^{\kappa+1} \in \mathcal{D}_{p \times q, \kappa+1}$ and Proposition 3.5, we have $(s_j)_{j=0}^{2n} \in \mathcal{D}_{p \times q, 2n}$ which, in view of Remark 5.3, implies $\Xi_{n, 2n} = 0_{(n+1)p \times (n+1)q}$. Since $(s_j)_{j=0}^{2n} \in \mathcal{D}_{p \times q, 2n}$ holds, Proposition 3.8(a) implies $(s_j)_{j=0}^{2n} \in \tilde{\mathcal{D}}_{p \times q, 2n}$. Thus, Lemma 9.1 and Remark 7.3 yield (9.1). Since $s_j^{[1, \alpha]} = AA^\dagger t_j A^\dagger A$ holds true for all $j \in \mathbb{Z}_{0, \kappa}$, we have, in view of (2.8) and (2.9), furthermore $-\alpha H_{n-1}^{[1, \alpha]} + K_{n-1}^{[1, \alpha]} = [I_n \otimes (AA^\dagger)](-\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)})[I_n \otimes (A^\dagger A)]$. Using $s_0 = A$, we thus get

$$\begin{aligned} & \text{diag} \left[A, \left[I_n \otimes (AA^\dagger) \right] (-\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)}) \left[I_n \otimes (A^\dagger A) \right] \right] \\ &= \text{diag}[s_0, -\alpha H_{n-1}^{[1, \alpha]} + K_{n-1}^{[1, \alpha]}] = \text{diag}[s_0, -\alpha H_{n-1}^{[1, \alpha]} + K_{n-1}^{[1, \alpha]}] + \Xi_{n, 2n} = \mathbf{D}_n H_n \mathbb{D}_n. \end{aligned}$$

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Since the matrices \mathbf{D}_n and \mathbb{D}_n are invertible, then (10.2) and (10.3) follow.

Now suppose $p = q$. In view of Remark 7.15 we obtain then $\det(\mathbf{D}_n H_n \mathbb{D}_n) = \det H_n$. Furthermore, a straightforward calculation shows that

$$\begin{aligned} \det \left(\text{diag} \left[A, \left[I_n \otimes (AA^\dagger) \right] (-\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)}) \left[I_n \otimes (A^\dagger A) \right] \right] \right) \\ = \det(A) \det(-\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)}). \end{aligned}$$

Hence, $\det H_n = \det(A) \det(-\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)})$ follows.

(c) In view of (7.5) and Remark 7.15, the matrices $\mathbf{D}_n^{[+, \alpha]}$ and $\mathbb{D}_n^{[+, \alpha]}$ are invertible. Because of $(s_j)_{j=0}^{\kappa+1} \in \mathcal{D}_{p \times q, \kappa+1}$ and Proposition 3.5, we have $(s_j)_{j=0}^{2n+1} \in \mathcal{D}_{p \times q, 2n+1}$, which, in view of Remark 5.3, implies $\Xi_{n, 2n+1} = 0_{(n+1)p \times (n+1)q}$. Since $(s_j)_{j=0}^{2n+1} \in \mathcal{D}_{p \times q, 2n+1}$ holds, Proposition 3.8(a) implies $(s_j)_{j=0}^{2n+1} \in \tilde{\mathcal{D}}_{p \times q, 2n+1}$. Thus, Lemma 7.19(a) and Remark 7.3 yield $H_n^{[1, \alpha]} = \mathbf{D}_n^{[+, \alpha]} (H_{\alpha \triangleright n} - \Xi_{n, 2n+1}) \mathbb{D}_n^{[+, \alpha]}$. Since $s_j^{[1, \alpha]} = AA^\dagger t_j A^\dagger A$ holds true for all $j \in \mathbb{Z}_{0, \kappa}$, we have, in view of (2.8), furthermore $H_n^{[1, \alpha]} = [I_{n+1} \otimes (AA^\dagger)] H_n^{(t)} [I_{n+1} \otimes (A^\dagger A)]$. Thus, we get

$$\begin{aligned} \left[I_{n+1} \otimes (AA^\dagger) \right] H_n^{(t)} \left[I_{n+1} \otimes (A^\dagger A) \right] &= H_n^{[1, \alpha]} = \mathbf{D}_n^{[+, \alpha]} (H_{\alpha \triangleright n} - \Xi_{n, 2n+1}) \mathbb{D}_n^{[+, \alpha]} \\ &= \mathbf{D}_n^{[+, \alpha]} H_{\alpha \triangleright n} \mathbb{D}_n^{[+, \alpha]}. \end{aligned}$$

Since the matrices $\mathbf{D}_n^{[+, \alpha]}$ and $\mathbb{D}_n^{[+, \alpha]}$ are invertible, then (10.4) and (10.5) follow.

Now suppose $p = q$. In view of (7.5) and Remark 7.15, we obtain then the equation $\det(\mathbf{D}_n^{[+, \alpha]} H_{\alpha \triangleright n} \mathbb{D}_n^{[+, \alpha]}) = \det H_{\alpha \triangleright n}$. Furthermore, we easily get

$$\det \left(\left[I_{n+1} \otimes (AA^\dagger) \right] H_n^{(t)} \left[I_{n+1} \otimes (A^\dagger A) \right] \right) = (\det A)(\det A)^\dagger \det H_n^{(t)}.$$

Hence, $\det H_{\alpha \triangleright n} = (\det A)(\det A)^\dagger \det H_n^{(t)}$ follows. \square

Corollary 10.12. *Let $\alpha \in \mathbb{C}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(t_j)_{j=0}^\kappa$ be a sequence of complex $p \times q$ matrices, and let A be a complex $p \times q$ matrix such that $\mathcal{N}(A) \subseteq \bigcap_{j=0}^\kappa \mathcal{N}(t_j)$ and $\bigcup_{j=0}^\kappa \mathcal{R}(t_j) \subseteq \mathcal{R}(A)$. Denote by $(s_j)_{j=0}^{\kappa+1}$ the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$. Then:*

(a) $H_0 = A$.

(b) Let $n \in \mathbb{N}$ with $2n - 1 \leq \kappa$. Then the matrices \mathbf{D}_n and \mathbb{D}_n are invertible and

$$H_n = \mathbf{D}_n^{-1} \left(\text{diag}[A, -\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)}] \right) \mathbb{D}_n^{-1}.$$

In particular, $\text{rank } H_n = \text{rank}(A) + \text{rank}(-\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)})$ and, in the case $p = q$, furthermore $\det H_n = \det(A) \det(-\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)})$.

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(c) Let $n \in \mathbb{N}_0$ with $2n \leq \kappa$. Then the matrices $\mathbf{D}_n^{[+, \alpha]}$ and $\mathbb{D}_n^{[+, \alpha]}$ are invertible and

$$H_{\alpha \triangleright n} = (\mathbf{D}_n^{[+, \alpha]})^{-1} H_n^{(t)} (\mathbb{D}_n^{[+, \alpha]})^{-1}. \quad (10.6)$$

In particular,

$$\text{rank } H_{\alpha \triangleright n} = \text{rank } H_n^{(t)} \quad (10.7)$$

and, in the case $p = q$, furthermore $\det H_{\alpha \triangleright n} = \det H_n^{(t)}$.

Proof. Using parts (c) and (b) of Remark A.1 we obtain $AA^\dagger t_j A^\dagger A = t_j$ for all $j \in \mathbb{Z}_{0, \kappa}$ which, in view of (2.8) and (2.9), implies $[I_{n+1} \otimes (AA^\dagger)] H_n^{(t)} [I_{n+1} \otimes (A^\dagger A)] = H_n^{(t)}$ for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$ and $[I_n \otimes (AA^\dagger)] (-\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)}) [I_n \otimes (A^\dagger A)] = -\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)}$ for all $n \in \mathbb{N}$ with $2n - 1 \leq \kappa$. The application of Proposition 10.11 yields then (a), (b), (10.6), and (10.7). For all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, from (10.6), (7.5), and Remark 7.15 we get in the case $p = q$ that $\det H_{\alpha \triangleright n} = \det H_n^{(t)}$. \square

Lemma 10.13. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(t_j)_{j=0}^\kappa$ be a sequence of Hermitian complex $q \times q$ matrices, and let A be a Hermitian complex $q \times q$ matrix. Then the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$ is a sequence of Hermitian complex $q \times q$ matrices.

Proof. Denote by $(s_j)_{j=0}^{\kappa+1}$ the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$ and let the sequence $(r_j)_{j=0}^{\kappa+1}$ be given by $r_j := s_j^{[\sharp, \alpha]}$ for all $j \in \mathbb{Z}_{0, \kappa+1}$. From Lemma 10.5 we infer then $r_j^\sharp = s_j^{[+, \alpha]}$ for all $j \in \mathbb{Z}_{0, \kappa+1}$ and, furthermore, $r_0 = A^\dagger$ and $r_j = -A^\dagger t_{j-1} A^\dagger$ for all $j \in \mathbb{Z}_{1, \kappa+1}$. In view of Remark A.1(a), we have $(A^\dagger)^* = (A^*)^\dagger = A^\dagger$. We then conclude that $r_j^* = r_j$ holds true for all $j \in \mathbb{Z}_{0, \kappa+1}$. By virtue of [26, Corollary 5.17], then $(r_j^\sharp)^* = r_j^\sharp$ for all $j \in \mathbb{Z}_{0, \kappa+1}$. Remark 4.3(e) yields then $s_j^* = s_j$ for all $j \in \mathbb{Z}_{0, \kappa+1}$. \square

The final investigations in this section are aimed at determining conditions which ensure that the first inverse α -S-transform belongs to the class $\mathcal{K}_{q, \kappa+1, \alpha}^\geq$ or one of its prominent subclasses.

Proposition 10.14. Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(t_j)_{j=0}^\kappa \in \mathcal{K}_{q, \kappa, \alpha}^\geq$, and let $A \in \mathbb{C}_{\geq}^{q \times q}$. Then the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$ belongs to $\mathcal{K}_{q, \kappa+1, \alpha}^\geq$.

Proof. Denote by $(s_j)_{j=0}^{\kappa+1}$ the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$. It is sufficient to check that, for all $n \in \mathbb{N}_0$ with $2n \leq \kappa + 1$, the matrix H_n is non-negative Hermitian and that, for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, the matrix $H_{\alpha \triangleright n}$ is non-negative Hermitian as well. According to Proposition 10.11(a) and $A \in \mathbb{C}_{\geq}^{q \times q}$, we have $H_0 = A \in \mathbb{C}_{\geq}^{q \times q}$. Now we consider an arbitrary $n \in \mathbb{N}$ with $2n \leq \kappa + 1$. Because of Proposition 10.11(b) the matrices \mathbf{D}_n and \mathbb{D}_n are invertible and (10.2) holds true. From Lemma 2.3(a) we know that $(t_j)_{j=0}^\kappa$ is a sequence of Hermitian complex $q \times q$ matrices. Taking additionally into account $A \in \mathbb{C}_{\geq}^{q \times q}$ and Lemma 10.13, we see then that $(s_j)_{j=0}^{\kappa+1}$ is

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a sequence of Hermitian complex $q \times q$ matrices. In view of Lemma 7.16, hence $\mathbf{D}_n^* = \mathbb{D}_n$. Using Remark A.1(a) and $A \in \mathbb{C}_{\geq}^{q \times q}$, we obtain $(AA^\dagger)^* = A^\dagger A$. Thus, (10.2) implies

$$H_n = \mathbf{D}_n^{-1} \left(\text{diag} \left[A, \left[I_n \otimes (AA^\dagger) \right] (-\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)}) \left[I_n \otimes (AA^\dagger) \right]^* \right] \right) \mathbf{D}_n^{-*}. \quad (10.8)$$

Using Remark 2.1, we get $(t_j)_{j=0}^{2n-1} \in \mathcal{K}_{q,2n-1,\alpha}^{\geq}$, which, in view of (1.3) and (1.5), implies $(-\alpha t_j + t_{j+1})_{j=0}^{2n-2} \in \mathcal{H}_{q,2n-2}^{\geq}$. Taking into account (2.8) and (2.9), we see then that $-\alpha H_{n-1}^{(t)} + K_{n-1}^{(t)}$ is a non-negative Hermitian matrix. In view of $A \in \mathbb{C}_{\geq}^{q \times q}$ and (10.8), it follows that H_n is a non-negative Hermitian matrix. Thus, for all $n \in \mathbb{N}_0$ with $2n \leq \kappa + 1$, the matrix H_n is proved to be non-negative Hermitian.

Finally, we consider an arbitrary $n \in \mathbb{N}_0$ with $2n \leq \kappa$. Because of Proposition 10.11(c), the matrices $\mathbf{D}_n^{[+, \alpha]}$ and $\mathbb{D}_n^{[+, \alpha]}$ are invertible and (10.4) holds true. We know that $(s_j)_{j=0}^{\kappa+1}$ is a sequence of Hermitian complex $q \times q$ matrices. Because of Remark 4.3(e), we see then that $(s_j^{[+, \alpha]})_{j=0}^{\kappa+1}$ is a sequence of Hermitian complex $q \times q$ matrices. In view of (7.5) and Lemma 7.16, hence $(\mathbf{D}_n^{[+, \alpha]})^* = \mathbb{D}_n^{[+, \alpha]}$. Taking additionally into account $A^\dagger A = (AA^\dagger)^*$, thus

$$H_{\alpha \triangleright n} = (\mathbf{D}_n^{[+, \alpha]})^{-1} \left[I_{n+1} \otimes (AA^\dagger) \right] H_n^{(t)} \left[I_{n+1} \otimes (AA^\dagger) \right]^* (\mathbf{D}_n^{[+, \alpha]})^{-*} \quad (10.9)$$

follows. Using Remark 2.1, we get $(t_j)_{j=0}^{2n} \in \mathcal{K}_{q,2n,\alpha}^{\geq}$, which, in view of (1.4), implies $(t_j)_{j=0}^{2n} \in \mathcal{H}_{q,2n}^{\geq}$. From (2.8) we see then that $H_n^{(t)}$ is a non-negative Hermitian matrix. Hence, (10.9) implies that $H_{\alpha \triangleright n}$ is a non-negative Hermitian matrix. Thus, for all $n \in \mathbb{N}_0$ with $2n \leq \kappa$, the matrix $H_{\alpha \triangleright n}$ is proved to be non-negative Hermitian. \square

Proposition 10.15. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(t_j)_{j=0}^{\kappa} \in \mathcal{K}_{q,\kappa,\alpha}^{\geq, e}$, and let $A \in \mathbb{C}_{\geq}^{q \times q}$. Then the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^{\kappa}, A]$ belongs to $\mathcal{K}_{q,\kappa+1,\alpha}^{\geq, e}$.*

Proof. In the case $\kappa = +\infty$, the assertion immediately follows from Proposition 10.14. Now let $\kappa \in \mathbb{N}_0$. Then there exists a complex $q \times q$ matrix $t_{\kappa+1}$ such that $(t_j)_{j=0}^{\kappa+1} \in \mathcal{K}_{q,\kappa+1,\alpha}^{\geq}$. From Proposition 10.14 we know then that the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^{\kappa+1}, A]$ belongs to $\mathcal{K}_{q,\kappa+2,\alpha}^{\geq}$. In view of Remark 10.2, the proof is complete. \square

Our next theme is to study the first inverse α -S-transform under the view of right α -Stieltjes parametrization. The two following results can be considered in some sense as inverse ones with respect to Theorem 9.26.

Theorem 10.16. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, let $(t_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq}$, and let $A \in \mathbb{C}_{\geq}^{q \times q}$ be such that $\mathcal{N}(A) \subseteq [\mathcal{N}(t_0)] \cap [\mathcal{N}(t_m)]$. Denote by $(s_j)_{j=0}^{m+1}$ the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^m, A]$ and by $(T_j)_{j=0}^m$ and $(Q_j)_{j=0}^{m+1}$ the right α -Stieltjes parametrizations of $(t_j)_{j=0}^m$ and $(s_j)_{j=0}^{m+1}$, respectively. Then $Q_0 = A$ and $Q_j = T_{j-1}$ for all $j \in \mathbb{Z}_{1,m+1}$.*

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Proof. Because of Definition 2.8, (2.11), and Definition 10.1, we get $Q_0 = L_0 = s_0 = A$. According to Proposition 10.14, the sequence $(s_j)_{j=0}^{m+1}$ belongs to $\mathcal{K}_{q,m+1,\alpha}^{\geq}$. By virtue of Lemma 2.3(a), we have $t_0^* = t_0$ and $t_m^* = t_m$, which, in view of $A \in \mathbb{C}_{\geq}^{q \times q}$ and $\mathcal{N}(A) \subseteq [\mathcal{N}(t_0)] \cap [\mathcal{N}(t_m)]$, implies $[\mathcal{R}(t_0)] \cup [\mathcal{R}(t_m)] \subseteq \mathcal{R}(A)$. Proposition 3.8(c) yields furthermore $(t_j)_{j=0}^m \in \tilde{\mathcal{D}}_{q \times q, m}$. Taking additionally into account Definitions 3.7 and 3.3, we conclude that $\mathcal{N}(A) \subseteq \bigcap_{j=0}^m \mathcal{N}(t_j)$ and $\bigcup_{j=0}^m \mathcal{R}(t_j) \subseteq \mathcal{R}(A)$. From Lemma 10.6(b) we see then that $(t_j)_{j=0}^m$ is exactly the first α -S-transform of $(s_j)_{j=0}^{m+1}$. The application of Theorem 9.25 with $k = 1$ yields then $T_m = Q_{m+1} - \epsilon_{m+1,0}$ (where $\epsilon_{m+1,0}$ is given via (9.10)) and, in the case $m \geq 1$, furthermore $T_l = Q_{1+l}$ for all $l \in \mathbb{Z}_{0,m-1}$. Remark 10.3 yields $(s_j)_{j=0}^{m+1} \in \mathcal{D}_{q \times q, m+1}$. Taking into account (9.10), Definitions 8.1 and 3.3, and parts (c) and (b) of Remark A.1, we obtain then $\epsilon_{m+1,0} = 0_{q \times q}$. Thus, $T_l = Q_{1+l}$ for all $l \in \mathbb{Z}_{0,m}$, which completes the proof. \square

Theorem 10.17. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(t_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$, and let $A \in \mathbb{C}_{\geq}^{q \times q}$ be such that $\mathcal{N}(A) \subseteq \mathcal{N}(t_0)$. Denote by $(s_j)_{j=0}^{\kappa+1}$ the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$ and by $(T_j)_{j=0}^\kappa$ and $(Q_j)_{j=0}^{\kappa+1}$ the right α -Stieltjes parametrization of $(t_j)_{j=0}^\kappa$ and $(s_j)_{j=0}^{\kappa+1}$, respectively. Then $Q_0 = A$ and $Q_j = T_{j-1}$ for all $j \in \mathbb{Z}_{1,\kappa+1}$.*

Proof. Because of Definition 2.8, (2.11), and Definition 10.1, we have $Q_0 = L_0 = s_0 = A$. According to Proposition 10.15, the sequence $(s_j)_{j=0}^{\kappa+1}$ belongs to $\mathcal{K}_{q,\kappa+1,\alpha}^{\geq,e}$. Obviously, we have $(t_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq}$. Because of Lemma 2.3(a), hence $t_0^* = t_0$, which, in view of $A \in \mathbb{C}_{\geq}^{q \times q}$ and $\mathcal{N}(A) \subseteq \mathcal{N}(t_0)$, implies $\mathcal{R}(t_0) \subseteq \mathcal{R}(A)$. Proposition 3.8(a) yields furthermore $(t_j)_{j=0}^\kappa \in \mathcal{D}_{q \times q, \kappa}$. From Proposition 10.8 we see then that $(t_j)_{j=0}^\kappa$ is exactly the first α -S-transform of $(s_j)_{j=0}^{\kappa+1}$. The application of Theorem 9.26 with $k = 1$ yields then $T_l = Q_{1+l}$ for all $l \in \mathbb{Z}_{0,\kappa}$, which completes the proof. \square

Proposition 10.18. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(t_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^>$, and let $A \in \mathbb{C}_{>}^{q \times q}$. Then the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$ belongs to $\mathcal{K}_{q,\kappa+1,\alpha}^>$.*

Proof. From Proposition 3.8(d) we obtain $(t_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq,e}$. Since A belongs to $\mathbb{C}_{>}^{q \times q}$, we have $A \in \mathbb{C}_{\geq}^{q \times q}$ and $\det A \neq 0$. In particular, $\mathcal{N}(A) \subseteq \mathcal{N}(t_0)$. Denote by $(s_j)_{j=0}^{\kappa+1}$ the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$. From Theorem 10.17 we get then $Q_0 = A$ and $Q_j = T_{j-1}$ for all $j \in \mathbb{Z}_{1,\kappa+1}$, where $(T_j)_{j=0}^\kappa$ and $(Q_j)_{j=0}^{\kappa+1}$ denote the right α -Stieltjes parametrizations of $(t_j)_{j=0}^\kappa$ and $(s_j)_{j=0}^{\kappa+1}$, respectively. According to Theorem 2.11(c), we have $T_l \in \mathbb{C}_{>}^{q \times q}$ for all $l \in \mathbb{Z}_{0,\kappa}$. In view of $A \in \mathbb{C}_{>}^{q \times q}$, thus $Q_j \in \mathbb{C}_{>}^{q \times q}$ for all $j \in \mathbb{Z}_{0,\kappa+1}$ which, in view of Theorem 2.11(c), implies $(s_j)_{j=0}^{\kappa+1} \in \mathcal{K}_{q,\kappa+1,\alpha}^>$. \square

Proposition 10.19. *Let $\alpha \in \mathbb{R}$, let $m \in \mathbb{N}_0$, let $(t_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,cd}$, and let $A \in \mathbb{C}_{\geq}^{q \times q}$ be such that $\mathcal{N}(A) \subseteq \mathcal{N}(t_0)$ in the case $m \geq 2$. Then the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^m, A]$ belongs to $\mathcal{K}_{q,m+1,\alpha}^{\geq,cd}$.*

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Proof. Because of Proposition 3.8(d), we have $(t_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq,e}$. The application of Proposition 10.15 yields then $(s_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\geq,e}$, where $(s_j)_{j=0}^{m+1}$ denotes the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^m, A]$. In particular, $(t_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq}$ and $(s_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\geq}$. We show now that $Q_{m+1} = 0_{q \times q}$ holds true, where $(Q_j)_{j=0}^{m+1}$ denotes the right α -Stieltjes parametrization of $(s_j)_{j=0}^{m+1}$.

First we consider the case $m = 0$. Because of (2.3) then $(t_j)_{j=0}^0$ belongs to $\mathcal{H}_{q,0}^{\geq,cd}$ which, in view of (2.11) and (2.2), implies $t_0 = L_0^{(t)} = 0_{q \times q}$. From Lemma 10.4 we obtain hence $s_0 = A$ and $s_1 = \alpha s_0 + AA^\dagger t_0 A^\dagger s_0^{[+, \alpha]} = \alpha A$. In view of Definition 2.8, (2.16), (2.11), and (1.3), this implies $Q_1 = L_{\alpha \triangleright 0} = s_{\alpha \triangleright 0} = -\alpha s_0 + s_1 = -\alpha A + \alpha A = 0_{q \times q}$.

Now we consider the case $m = 1$. Because of (1.3) and (2.4), then the sequence $(u_j)_{j=0}^0$ given by $u_0 := -\alpha t_0 + t_1$ belongs to $\mathcal{H}_{q,0}^{\geq,cd}$ which, in view of (2.11) and (2.2), implies $u_0 = L_0^{(u)} = 0_{q \times q}$. Thus, $t_1 = \alpha t_0$. Using Definition 4.1 and Lemma 10.4, then $s_0 = A$,

$$s_1 = \alpha s_0 + AA^\dagger t_0 A^\dagger s_0^{[+, \alpha]} = \alpha s_0 + AA^\dagger t_0 A^\dagger s_0 = \alpha A + AA^\dagger t_0 A^\dagger A,$$

and

$$\begin{aligned} s_2 &= \alpha s_1 + AA^\dagger (t_1 A^\dagger s_0^{[+, \alpha]} + t_0 A^\dagger s_1^{[+, \alpha]}) = \alpha s_1 + AA^\dagger [t_1 A^\dagger s_0 + t_0 A^\dagger (-\alpha s_0 + s_1)] \\ &= \alpha s_1 + AA^\dagger (\alpha t_0 A^\dagger s_0 - \alpha t_0 A^\dagger s_0 + t_0 A^\dagger s_1) = \alpha s_1 + AA^\dagger t_0 A^\dagger s_1 \\ &= \alpha (\alpha A + AA^\dagger t_0 A^\dagger A) + AA^\dagger t_0 A^\dagger (\alpha A + AA^\dagger t_0 A^\dagger A) \\ &= \alpha^2 A + 2\alpha AA^\dagger t_0 A^\dagger A + AA^\dagger t_0 A^\dagger AA^\dagger t_0 A^\dagger A \\ &= (\alpha A + AA^\dagger t_0 A^\dagger A) A^\dagger (\alpha A + AA^\dagger t_0 A^\dagger A) = s_1 s_0^\dagger s_1. \end{aligned}$$

In view of Definition 2.8, (2.11), (2.7), (1.1), and (1.3), this implies $Q_2 = L_1 = s_2 - z_{1,1} H_0^\dagger y_{1,1} = s_2 - s_1 s_0^\dagger s_1 = 0_{q \times q}$.

Finally, we consider the case $m \geq 2$. From Theorem 10.17 we obtain then $Q_{m+1} = T_m$, where $(T_j)_{j=0}^m$ denotes the right α -Stieltjes parametrization of $(t_j)_{j=0}^m$. According to [21, Proposition 5.3], we have $T_m = 0_{q \times q}$ and hence $Q_{m+1} = 0_{q \times q}$.

Thus, in each case, $Q_{m+1} = 0_{q \times q}$ holds true, which, in view of $(s_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\geq}$ and [21, Proposition 5.3], implies $(s_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\geq,cd}$. \square

The following example shows that, in the case $m \geq 2$, the assumption $\mathcal{N}(A) \subseteq \mathcal{N}(t_0)$ in Proposition 10.19 cannot be omitted.

Example 10.20. Let $q := 2$, let $\alpha := -1$, let $t_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, let $t_1 := \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, let $t_2 := \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, and let $A := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. According to (2.8), then

$$H_1^{(t)} = \begin{bmatrix} t_0 & t_1 \\ t_1 & t_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}^* \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \in \mathbb{C}_{\geq}^{4 \times 4},$$

which implies $(t_j)_{j=0}^2 \in \mathcal{H}_{q,2}^{\geq}$. Obviously, $-\alpha t_0 + t_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \in \mathbb{C}_{\geq}^{2 \times 2}$, which, in view of (2.8), implies $(-\alpha t_j + t_{j+1})_{j=0}^0 \in \mathcal{H}_{q,0}^{\geq}$. According to (1.3) and (1.4), thus $(t_j)_{j=0}^2 \in$

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$\mathcal{K}_{q,2,\alpha}^{\geq}$. Because of (2.11), (2.7), and (2.8), furthermore $L_1^{(t)} = t_2 - z_{1,1}^{(t)}(H_0^{(t)})^\dagger y_{1,1}^{(t)} = t_2 - t_1 t_0^\dagger t_1 = 0_{2 \times 2}$. By (2.2), hence $(t_j)_{j=0}^2 \in \mathcal{H}_{q,2}^{\geq, \text{cd}}$. According to (2.3), then $(t_j)_{j=0}^2 \in \mathcal{K}_{q,2,\alpha}^{\geq, \text{cd}}$. Obviously, $A \in \mathbb{C}_{\geq}^{2 \times 2}$ and $\mathcal{N}(A) \not\subseteq \mathcal{N}(t_0)$. Denote by $(s_j)_{j=0}^3$ the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^2, A]$. Using $A^2 = A$, $A^\dagger = A$, $At_1A = A$, $At_2A = 2A$, and Definition 4.1, we obtain from Lemma 10.4 then $s_0 = A$,

$$\begin{aligned} s_1 &= \alpha s_0 + AA^\dagger t_0 A^\dagger s_0^{[+, \alpha]} = \alpha s_0 + AA^\dagger t_0 A^\dagger s_0 = -A + AA^\dagger \cdot I_2 \cdot A^\dagger A = 0_{2 \times 2}, \\ s_2 &= \alpha s_1 + AA^\dagger (t_1 A^\dagger s_0^{[+, \alpha]} + t_0 A^\dagger s_1^{[+, \alpha]}) = \alpha s_1 + AA^\dagger [t_1 A^\dagger s_0 + t_0 A^\dagger (-\alpha s_0 + s_1)] \\ &= -0_{2 \times 2} + AA^\dagger [t_1 A^\dagger A + I_2 \cdot A^\dagger (A + 0_{2 \times 2})] = 2A, \end{aligned}$$

and, consequently,

$$\begin{aligned} s_3 &= \alpha s_2 + AA^\dagger (t_2 A^\dagger s_0^{[+, \alpha]} + t_1 A^\dagger s_1^{[+, \alpha]} + t_0 A^\dagger s_2^{[+, \alpha]}) \\ &= \alpha s_2 + AA^\dagger [t_2 A^\dagger s_0 + t_1 A^\dagger (-\alpha s_0 + s_1) + t_0 A^\dagger (-\alpha s_1 + s_2)] \\ &= -2A + AA^\dagger [t_2 A^\dagger A + t_1 A^\dagger (A + 0_{2 \times 2}) + I_2 \cdot A^\dagger (0_{2 \times 2} + 2A)] = 3A. \end{aligned}$$

In view of (1.3), then $s_{\alpha \triangleright 0} = A$ and $s_{\alpha \triangleright 1} = 2A$, and $s_{\alpha \triangleright 2} = 5A$, which, in view of (2.16), (2.11), (2.17), (2.15), (2.7), and (2.8), implies $L_{\alpha \triangleright 1} = s_{\alpha \triangleright 2} - z_{\alpha \triangleright 1,1}(H_{\alpha \triangleright 0})^\dagger y_{\alpha \triangleright 1,1} = A \neq 0_{2 \times 2}$. By (2.2), hence $(s_{\alpha \triangleright j})_{j=0}^2 \notin \mathcal{H}_{q,2}^{\geq, \text{cd}}$. According to (2.4), then $(s_j)_{j=0}^3 \notin \mathcal{K}_{q,3,\alpha}^{\geq, \text{cd}}$.

Proposition 10.21. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $m \in \mathbb{Z}_{0,\kappa}$, let $(t_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq, \text{cd}, m}$, and let $A \in \mathbb{C}_{\geq}^{q \times q}$ be such that $\mathcal{N}(A) \subseteq \mathcal{N}(t_0)$ in the case $m \geq 2$. Then the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$ belongs to $\mathcal{K}_{q,\kappa+1,\alpha}^{\geq, \text{cd}, m+1}$.*

Proof. Because of (2.5), we have $(t_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq}$ and $(t_j)_{j=0}^m \in \mathcal{K}_{q,m,\alpha}^{\geq, \text{cd}}$. The application of Proposition 10.14 yields $(s_j)_{j=0}^{\kappa+1} \in \mathcal{K}_{q,\kappa+1,\alpha}^{\geq}$, where $(s_j)_{j=0}^{\kappa+1}$ denotes the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$. From Proposition 10.19 we obtain, in view of Remark 10.2, furthermore $(s_j)_{j=0}^{m+1} \in \mathcal{K}_{q,m+1,\alpha}^{\geq, \text{cd}}$. According to (2.5), we have then $(s_j)_{j=0}^{\kappa+1} \in \mathcal{K}_{q,\kappa+1,\alpha}^{\geq, \text{cd}, m+1}$. \square

Corollary 10.22. *Let $\alpha \in \mathbb{R}$, let $\kappa \in \mathbb{N}_0 \cup \{+\infty\}$, let $(t_j)_{j=0}^\kappa \in \mathcal{K}_{q,\kappa,\alpha}^{\geq, \text{cd}}$, and let $A \in \mathbb{C}_{\geq}^{q \times q}$ be such that $\mathcal{N}(A) \subseteq \mathcal{N}(t_0)$. Then the first inverse α -S-transform corresponding to $[(t_j)_{j=0}^\kappa, A]$ belongs to $\mathcal{K}_{q,\kappa+1,\alpha}^{\geq, \text{cd}}$.*

Proof. Use Proposition 10.19, (2.6), and Proposition 10.21. \square

A. Some facts from matrix theory

Remark A.1. Let A be a complex $p \times q$ matrix and let $m, n \in \mathbb{N}$. Then:

- (a) $(A^\dagger)^\dagger = A$, $(A^\dagger)^* = (A^*)^\dagger$, $\mathcal{N}(A^\dagger) = \mathcal{N}(A^*)$, and $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*)$.

- (b) If B is a complex $m \times q$ matrix, then $\mathcal{N}(A) \subseteq \mathcal{N}(B)$ if and only if $BA^\dagger A = B$.
- (c) If C is a complex $p \times n$ matrix, then $\mathcal{R}(C) \subseteq \mathcal{R}(A)$ if and only if $AA^\dagger C = C$.
- (d) If U is complex $m \times p$ matrix with $U^*U = I_p$ and if V is a complex $q \times n$ matrix with $VV^* = I_q$, then $(UAV)^\dagger = V^*A^\dagger U^*$.

Note that the sets $\mathfrak{L}_{q,n}$ and $\mathfrak{U}_{q,n}$ are defined in Notation 7.14.

Remark A.2. Let $n \in \mathbb{N}_0$. Then $\mathfrak{L}_{q,n}$ and $\mathfrak{U}_{q,n}$ are subgroups of the general linear group $\text{GL}((n+1)q, \mathbb{C})$ of all non-singular complex $(n+1)q \times (n+1)q$ matrices with $\mathfrak{L}_{q,n} \cap \mathfrak{U}_{q,n} = \{I_{(n+1)q}\}$. Furthermore, $\det A = 1$ for all $A \in \mathfrak{L}_{q,n} \cup \mathfrak{U}_{q,n}$, $\mathfrak{U}_{q,n} = \{L^* | L \in \mathfrak{L}_{q,n}\}$, and $\mathfrak{L}_{q,n} = \{U^* | U \in \mathfrak{U}_{q,n}\}$.

Remark A.3. Let $m, n \in \mathbb{N}_0$. In view of Notation 7.14, then $\text{diag}[L, M] \in \mathfrak{L}_{q,m+n+1}$ for all $L \in \mathfrak{L}_{q,m}$ and all $M \in \mathfrak{L}_{q,n}$ and $\text{diag}[U, V] \in \mathfrak{U}_{q,m+n+1}$ for all $U \in \mathfrak{U}_{q,m}$ and all $V \in \mathfrak{U}_{q,n}$.

Remark A.4. Let $n \in \mathbb{N}_0$, let $(A_j)_{j=0}^n$ and $(B_j)_{j=0}^n$ be sequences of complex $p \times q$ matrices, and let $E, V \in \mathfrak{L}_{p,n}$ and $F, W \in \mathfrak{U}_{q,n}$ be such that $E \cdot \text{diag}[A_0, A_1, \dots, A_n] \cdot F = V \cdot \text{diag}[B_0, B_1, \dots, B_n] \cdot W$. Then we easily see that $A_j = B_j$ holds true for all $j \in \mathbb{Z}_{0,n}$.

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